A finite dimensional approach to the strong Novikov conjecture

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Abstract

The aim of this paper is to introduce an approach to the (strong) Novikov conjecture based on continuous families of finite dimensional representations: this is partly inspired by ideas of Lusztig using the Atiyah-Singer families index theorem, and partly by Carlsson's deformation K–theory. Using this approach, we give new proofs of the strong Novikov conjecture in several interesting cases, including crystallographic groups and surface groups. The method presented here is relatively accessible compared with other proofs of the Novikov conjecture, and also yields some information about the K–theory and cohomology of representation spaces.

1 Introduction

The aim of this paper is to study the *strong Novikov conjecture* [16] for a finitely presented group Γ. If we assume that Γ has a finite classifying space $B\Gamma$, one version of this conjecture states that the analytic assembly map

$$\mu : K_*(B\Gamma) \to K_*(C^*(\Gamma))$$

is rationally injective; here the left hand side is the K–homology of $B\Gamma$ and the right hand side is the K–theory of the maximal group C^*-algebra of Γ. We give a definition of the analytic assembly map in Section 2 below. The strong Novikov conjecture implies the usual Novikov conjecture on homotopy invariance of higher signatures, as well as being closely related to several other famous conjectures.

A naive approach to proving this conjecture might proceed as follows. A finite dimensional unitary representation

$$\rho : \Gamma \to U(n) \tag{1}$$

of Γ defines a vector bundle E_ρ over $B\Gamma$ via a well-known balanced product construction. E_ρ defines an element $[E_\rho]$ of the K–theory group $K^*(B\Gamma)$ and thus a 'detecting homomorphism'

$$\rho_* : K_*(B\Gamma) \to \mathbb{Z} \tag{2}$$

defined by pairing with $[E_\rho]$. As is well-known[1], ρ_* factors through the analytic assembly map μ; hence $\mu(x) \neq 0$ for any $x \in K_*(B\Gamma)$ such that there exists $\rho : \Gamma \to U(n)$ with $\rho_*(x) \neq 0$. Thus if one can find 'enough' representations to detect all of $K_*(B\Gamma)$, one would have proved the strong Novikov conjecture.

Unfortunately, this approach will not work: the bundles E_ρ are flat, so Chern-Weil theory tells us that any 'detecting homomorphism' as in line (2) above is rationally trivial on reduced K–homology. One possible way to salvage the idea in the paragraph above is to use infinite dimensional representations. This led to the *Fredholm representations* of Miscenko [18], and subsequently to Kasparov's *KK-theory* [16]; both of these, and the closely related approach to the Novikov conjecture through the *Baum-Connes conjecture* [7] have proved enormously fruitful.

In this paper, we suggest a different approach. The central idea is not to consider a single representation as in line (1) above, but instead a continuous *family* of representations

$$\rho : X \to \mathrm{Hom}(\Gamma, U(n))$$

parametrized by a topological space X. Such a family defines a bundle E_ρ over $X \times B\Gamma$ and thus a detecting homomorphism

$$\rho_* : K_*(B\Gamma) \to K^*(X)$$

from the K–homology of $B\Gamma$ to the K–theory of X via slant product with $[E_\rho] \in K^*(B\Gamma \times X)$. This ρ_* still factors through the analytic assembly map. It is a central result of this paper that for many interesting groups, there *are* enough detecting homomorphisms of this type to 'see' all of $K_*(B\Gamma)$. The strong Novikov conjecture follows. In addition, we obtain some information about the K–theory and cohomology of the representation varieties $\mathrm{Hom}(\Gamma, U(n))$, which have received a good deal of attention recently from a number of authors (see, for instance, [1, 5] for free abelian groups, and [23] for surface groups).

One precursor for these ideas is Lusztig's thesis [17], where the Atiyah-Singer index theorem for families [4] was used to study a version of the analytic assembly map for $\mathbb{Z}^n$. This is related to *Mukai duality* for the n-torus [19]. We were initially inspired by Carlsson's *deformation K–theory*, which in some sense develops related ideas in homotopy theory and algebraic K–theory. Carlsson associates to Γ a *spectrum* (in the sense of stable homotopy theory) $K^{\mathrm{def}}(\Gamma)$, built from the (topological) category of finite dimensional unitary representations of Γ (see Ramras [20] for a description of this construction). The homotopy groups of this spectrum can be described in terms of spherical families of representations, and the topological Atiyah–Segal map

$$\pi_* K^{\mathrm{def}}(\Gamma) \longrightarrow K^{-*}(B\Gamma)$$

considered in [6] might be viewed as a sort of dual to the analytic assembly map. Our results show that rational surjectivity of this map implies rational

[1] It is also a special case of Proposition 4.1 below, to which we refer the reader for a proof.

injectivity of the analytic assembly map (however, from the perspective of the Novikov conjecture, there is no reason to restrict attention to spherical families, and indeed we gain some ground by allowing our families to be parametrized by arbitrary spaces X).

Our results on the Novikov conjecture are not new: the strong Novikov conjecture is known for a huge class of groups, and we are not able to add any new cases (in fact, there is no group which is known to lie outside the scope of current results on the problem). However, the methods are new, and we hope intrinsically interesting. Moreover, they are almost all elementary, and we have aimed to keep the paper as self-contained as possible, avoiding the use of complicated general theories wherever we can. We hope this makes the paper a good introduction to aspects of the theory, both for C^*-algebraists who know a little topology, and topologists who know a little about C^*-algebras.

2 Slant products and assembly in analytic K–theory

In this section we use Paschke duality (as refined by Higson [13] and Higson–Roe [14, Chapter 5]) to give a concrete description of one of the slant products in operator K–theory. This slant product was perhaps first given an analytic definition by Atiyah and Singer via their families index theorem [4], and subsequently by Kasparov in the much broader context of his bivariant KK-theory (see for example [16]); our approach is perhaps simpler and more direct than either of these, however. It is inspired by (but not the same as) the slant product briefly discussed in [14, Exercise 9.8.9].

We then use this slant product and the so-called *Miscenko bundle* to give a relatively straightforward approach to the analytic assembly map

$$\mu\colon\ K_*(B\Gamma) \to K_*(C^*(\Gamma))$$

in the case that Γ is a discrete group admitting a finite classifying space $B\Gamma$. All of this could of course be done using Kasparov's bivariant KK-theory [16], but our approach seems simpler and more direct.

See [14, Chapters 4 and 5] for background information on analytic K–theory and the Paschke duality approach to K–homology theory used in what follows.

Definition 2.1 ([14], Chapter 5)**.** Let A be a C^*-algebra. A representation of A on a Hilbert space $\mathcal{H}$ is said to be *nondegenerate* if $\{a\xi \mid a \in A,\ \xi \in \mathcal{H}\}$ is dense in $\mathcal{H}$ (for example, if the representation is unital).

A representation of A on $\mathcal{H}$ is said to be *ample* if it is nondegenerate and no non-zero element in A acts as a compact operator on $\mathcal{H}$.

Let now $\tilde{A}$ be the unitization of A (even if A is already unital, in which case $\tilde{A} \cong A \oplus \mathbb{C}$) and fix an ample representation of $\tilde{A}$. The *dual* of A is the C^*-algebra

$$\mathcal{D}(A) := \{T \in \mathcal{B}(\mathcal{H}) \mid [T, a] \in \mathcal{K}(\mathcal{H}) \text{ for all } a \in \tilde{A}\},$$

i.e. the set of operators on $\mathcal{H}$ that commute with $\tilde{A}$ up to compact operators. It does not depend on the choice of ample representation up to non-canonical isomorphism. Moreover, the K–theory groups of $\mathcal{D}(A)$, $K_*(\mathcal{D}(A))$, do not depend on the choice of ample representation up to canonical isomorphism. For the purpose of this piece, we follow [14, Definition 5.2.7], and define the i^{th} K–homology group of A to be

$$K^i(A) := K_{1-i}(\mathcal{D}(A)).$$

Definition 2.2. Let A and B be C^*-algebras, and let $\mathcal{H}_A$ and $\mathcal{H}_B$ be ample representations of $\tilde{A}$ and $\tilde{B}$ respectively. Let the spatial tensor product $A \otimes B$ be represented on $\mathcal{H}_A \otimes \mathcal{H}_B$ in the natural way. Consider also the C^*-algebras

$$A \otimes \mathcal{K} := A \otimes \mathcal{K}(\mathcal{H}_B) \subseteq \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$$

and

$$A \otimes \mathcal{B} := A \otimes \mathcal{B}(\mathcal{H}_B) \subseteq \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$$

Define a function

$$\sigma = \sigma^{A,B} \colon \ (A \otimes B) \otimes \mathcal{D}(B) \to \frac{A \otimes \mathcal{B}}{A \otimes \mathcal{K}}$$

(where we consider $\mathcal{D}(B)$ as defined using $\mathcal{H}_B$) by the formula

$$\sigma : (a \otimes b) \otimes T \mapsto a \otimes bT; \tag{3}$$

note moreover that if $a \in A$, $b \in B$ and $T \in \mathcal{D}(B)$, then the elements $a \otimes b$ and $1 \otimes T$ in $\mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ commute up to elements of $A \otimes \mathcal{K}$, whence it follows that σ is actually a $*$-homomorphism.[2]

The $*$-homomorphism σ thus induces a map on K–theory that fits into the composition

$$K_i(A \otimes B) \otimes K_j(\mathcal{D}(B)) \to K_{i+j}(A \otimes B \otimes \mathcal{D}(B)) \stackrel{\sigma_*}{\to} K_{i+j}\Big(\frac{A \otimes \mathcal{B}}{A \otimes \mathcal{K}}\Big), \tag{4}$$

where the first map is the usual (external) product in operator K–theory ([14, Section 4.7]). Now, the definition of K–homology in terms of dual algebras yields $K_i(\mathcal{D}(B)) = K^{1-i}(B)$. Moreover, using that $K_*(A \otimes \mathcal{B}) = 0$ (this follows from an easy Eilenberg swindle argument, just as for $\mathcal{B}$ itself) and the long exact sequence in K–theory we have natural isomorphisms

$$K_i\Big(\frac{A \otimes \mathcal{B}}{A \otimes \mathcal{K}}\Big) \cong K_{i-1}(A \otimes \mathcal{K}) \cong K_{i-1}(A).$$

Thus line (4) is equivalent to a map

$$K_i(A \otimes B) \otimes K^j(B) \to K_{(i+(1-j))-1}(A) = K_{i-j}(A).$$

We call the map in the line above the *slant product* in operator K–theory. If $x \in K_i(A \otimes B)$ and $y \in K^j(B)$, we denote their slant product by $x/y \in K_{i-j}(A)$.

[2] It would perhaps be more natural to use the stable multiplier algebra $M(A \otimes \mathcal{K})$ where we have used $A \otimes \mathcal{B}$; the latter is certainly good enough for our purposes, however, and seems to have functoriality properties that are somewhat simpler to analyze. The fact '$K_*(A \otimes \mathcal{B}) = 0$' is also significantly easier than '$K_*(M(A \otimes \mathcal{K})) = 0$', which is helpful for us.

Example 2.3. Say in the above that $A = \mathbb{C}$, so that $A \otimes B$ is canonically isomorphic to B, and the slant product reduces to a pairing

$$K_i(B) \otimes K^j(B) \to K_{i-j}(\mathbb{C}) \cong \begin{cases} \mathbb{Z} & i = j \bmod 2 \\ 0 & \text{otherwise} \end{cases}.$$

This pairing can be identified with the usual pairing between K–theory and K–homology as we now explain. Assume throughout for simplicity that B is unital (which is in any case all we will need).

Assume also, at least for the moment, that $i = j = 0$. It will suffice to show that the pairing above agrees with the usual pairing between K–homology and K–theory when $[p] \in K_0(B)$ is a class represented by some projection $p \in M_n(B)$, and $[u] \in K^0(B)$ is represented by some unitary $u \in \mathcal{D}(B)$.

Now, according to [14, Proposition 4.8.3], the image of the element

$$[p] \otimes [u] \in K_0(B) \otimes K_0(\mathcal{D}(B))$$

under the product map to $K_0(B \otimes \mathcal{D}(B))$ can be represented by the unitary

$$p \otimes u + (1-p) \otimes 1 \in M_n(B) \otimes \mathcal{D}(B) \cong M_n(B \otimes \mathcal{D}(B)).$$

Let $\mathcal{Q} = \mathcal{B}/\mathcal{K}$ denote the Calkin algebra and for $x \in \mathcal{B}$ write $\overline{x}$ for its image under the quotient map $\mathcal{B} \to \mathcal{Q}$. Write u_n for the element of $M_n(\mathcal{B})$ with all diagonal entries u, and all other entries zero. Then it is not hard to check that the natural extension of $\sigma^{\mathbb{C},B}$ to the matrix algebra $M_n(B \otimes \mathcal{D}(B))$ acts as follows:

$$\sigma^{\mathbb{C},B} \colon\ p \otimes u + (1-p) \otimes 1 \mapsto \overline{pu_n + (1-p)} \in \frac{M_n(\mathcal{B})}{M_n(\mathcal{K})} \cong M_n(\mathcal{Q}).$$

Using that p is a projection, and that p and u_n commute up to $M_n(\mathcal{K})$, we have that

$$\overline{pu_n + (1-p)} = \overline{p^2 u_n + (1-p)} = \overline{pu_n p + (1-p)},$$

whence the slant product of $[p]$ and $[u]$ is equal to the image of the class of $\overline{pu_n p + 1 - p}$ in $K_1(\mathcal{Q})$ under the boundary map

$$\partial\ :\ K_1(\mathcal{Q}) \to K_0(\mathcal{K}) \cong \mathbb{Z}.$$

In this special case, however, this boundary map is concretely realized by the formula

$$K_1(\mathcal{Q}) \to \mathbb{Z}, \quad [\overline{v}] \mapsto \mathrm{Index}(v)$$

(see for example [14, Proposition 4.8.8]); note that if $\overline{v} \in M_n(\mathcal{Q})$ is unitary, then $v \in M_n(\mathcal{B})$ is Fredholm by Atkinson's theorem, so this makes sense. Our conclusion, finally, is that the slant product is given by the integer

$$\mathrm{Index}(pu_n p + 1 - p)$$

which is the formula for the pairing between K–homology and K–theory from [14, Section 7.2]. The only other case of interest is $i = j = 1$; this works analogously, however, using the same product formula.

The following lemma, giving two simple naturality properties of the slant product, will be needed later.

Lemma 2.4. *1. The slant product*

$$K_i(A \otimes B) \otimes K^j(B) \to K_{i-j}(A)$$

is functorial in the sense that if $\phi\colon A \to C$ is a $*$-homomorphism, $x \in K_*(A \otimes B)$ and $y \in K^*(B)$, then

$$\phi_*(x/y) = ((\phi \otimes 1_B)_* x)/y$$

as elements of $K_*(C)$.

2. Let A, B, C be unital C^*-algebras, x be a class in $K_i(A \otimes B)$, y be a class in $K^j(B)$ and z be a class in $K_k(C)$. Then

$$(z \otimes x)/y = z \otimes (x/y)$$

as elements of $K_{i+k-j}(C \otimes A)$.

Proof. Look first at part (1). With notation as in Definition 2.2, note first that if

$$\phi \otimes 1_{\mathcal{B}} : \frac{A \otimes \mathcal{B}}{A \otimes \mathcal{K}} \to \frac{C \otimes \mathcal{B}}{C \otimes \mathcal{K}}$$

is the natural $*$-homomorphism induced by ϕ, then the definition of

$$\sigma^{A,B}\colon\ A \otimes B \otimes \mathcal{D}(B) \to \frac{A \otimes \mathcal{B}}{A \otimes \mathcal{K}}$$

implies that

$$\sigma^{C,B} \circ (\phi \otimes 1_B \otimes 1_{\mathcal{D}(B)}) = (\phi \otimes 1_{\mathcal{B}}) \circ \sigma^{A,B}. \tag{5}$$

It follows then from the definition of the slant product that, up to the isomorphism

$$K_i\Big(\frac{A \otimes \mathcal{B}}{A \otimes \mathcal{K}}\Big) \cong K_{i-1}(A) \tag{6}$$

(and similarly with A replaced by C), the K–theory element $\phi_*(x/y)$ is equal to

$$\begin{aligned}(\phi \otimes 1_{\mathcal{B}})_*(\sigma_*^{A,B}(x \otimes y)) &= \sigma_*^{C,B}((\phi \otimes 1_B \otimes 1_{\mathcal{D}(B)})_*(x \otimes y)) \\ &= \sigma_*^{C,B}((((\phi \otimes 1_B)_* x) \otimes y) \\ &= ((\phi \otimes 1_B)_* x)/y),\end{aligned}$$

where we have used line (5) in the first equality and naturality of the K–theory product in the second. Up to the isomorphism in line (6) again, this is exactly the statement of the lemma.

Part (2) is a simple consequence of the formula in line (3) above, and naturality properties of the K–theory product with respect to $*$-homomorphisms and boundary maps [14, Proposition 4.7.6]. ☐

The following K–theory class is important for the definition of assembly.

Definition 2.5. Let Γ be a (finitely presented) discrete group with finite classifying space $B\Gamma$, and let $C^*(\Gamma)$ denote the maximal group C^*-algebra for Γ. Let $E\Gamma$ be the universal covering space of $B\Gamma$. Then the *Miscenko bundle* for Γ, denoted M_Γ, is the bundle over $B\Gamma$ with fibres $C^*(\Gamma)$ defined as the quotient of the space $E\Gamma \times C^*(\Gamma)$ by the diagonal action

$$g \cdot (z, a) := (gz, u_g a),$$

where $u_g \in C^*(G)$ is the unitary element of this C^*-algebra corresponding to g.

Lemma 2.6. *The C^*-algebra*

$$C(B\Gamma, C^*(\Gamma)) \cong C^*(\Gamma) \otimes C(B\Gamma)$$

acts naturally on the right of the space of sections of the Miscenko bundle, and this space of sections is a finitely generated projective module over $C^*(\Gamma) \otimes C(B\Gamma)$.

In particular, the Miscenko bundle defines a class

$$[M_\Gamma] \in K_*(C^*(\Gamma) \otimes C(B\Gamma)).$$

Proof. We first define the $C^*(\Gamma) \otimes C(B\Gamma)$ module structure on the sections of M_Γ. Let $C_b(E\Gamma, C^*(\Gamma))$ denote the C^*-algebra of continuous bounded functions from $E\Gamma$ to $C^*(\Gamma)$, which admits a natural left–Γ action defined for $z \in E\Gamma$, $g \in \Gamma$ and $f \in C_b(E\Gamma, C^*(\Gamma))$ by

$$(g \cdot f)(z) := u_g f(g^{-1}z);$$

the space of sections of M_Γ then clearly identifies with the fixed point subalgebra $C_b(E\Gamma, C^*(\Gamma))^\Gamma$, consisting of Γ–equivariant maps. Moreover, if $\pi\colon E\Gamma \to B\Gamma$ is the canonical quotient, then the formula

$$(f \cdot h)(z) := f(z)h(\pi(z))$$

for $f \in C_b(E\Gamma, C^*(\Gamma))^\Gamma$ and $h \in C(B\Gamma, C^*(\Gamma))$ defines a right action of the algebra $C(B\Gamma, C^*(\Gamma))$ on the sections of the Miscenko bundle; we must show that this makes this space of sections into a finitely generated projective module over $C(B\Gamma, C^*(\Gamma))$.

Note then that the Miscenko bundle is locally trivial (as it is locally isomorphic to the bundle $E\Gamma \times C^*(\Gamma)$), so there exists a finite open cover of $B\Gamma$, say $\{U_1, ..., U_n\}$, such that the closure of each U_i is contained in some open set V_i over which the Miscenko bundle is trivial. Let $\{\phi_i\}_{i=1}^n$ be a partition of unity subordinate to $\{U_i\}_{i=1}^n$, and for each i let ψ_i be a function on $B\Gamma$ that is equal to 1 on U_i and vanishes outside V_i. For each i, let $\widetilde{U_i}$ and $\widetilde{V_i}$ be arbitrary choices of homeomorphic lifts of U_i, V_i respectively, and by abuse of notation identify functions supported in $\widetilde{U_i}$ and U_i, and functions supported on $\widetilde{V_i}$ and V_i, without further comment. Then the $C(B\Gamma, C^*(\Gamma))$-module map

$$\begin{aligned} \Phi\colon\ C_b(E\Gamma, C^*(\Gamma))^\Gamma &\to C(B\Gamma, C^*(\Gamma))^{\oplus n} \\ f &\mapsto \oplus_{i=1}^n (\phi_i f|_{\widetilde{U_i}}) \end{aligned}$$

includes $C_b(E\Gamma, C^*(\Gamma))^\Gamma$ as a submodule of the free module $C(B\Gamma, C^*(\Gamma))^{\oplus n}$, and is moreover split by the $C(B\Gamma, C^*(\Gamma))$-module map

$$\begin{aligned} \Psi\colon\ C(B\Gamma, C^*(\Gamma))^{\oplus n} &\to C_b(E\Gamma, C^*(\Gamma))^\Gamma \\ (f_i)_{i=1}^n &\mapsto \sum_{i=1}^n \psi_i f_i|_{V_i}; \end{aligned}$$

this shows that $C_b(E\Gamma, C^*(\Gamma))^\Gamma$ is finitely generated and projective as required. □

Definition 2.7. Let Γ, $B\Gamma$ be as in the previous definition. Then the *analytic assembly map* is the homomorphism

$$\mu\colon\ K_*(B\Gamma) \to K_*(C^*(\Gamma))$$

defined by taking the slant product with the class of the Miscenko bundle $[M_\Gamma]$, i.e.

$$\mu(x) = [M_\Gamma]/x$$

for all $x \in K_*(B\Gamma)$.

3 Families of representations and the class $\mathcal{FD}$

Throughout this section, we let Γ denote a (finitely presented) group with finite classifying space $B\Gamma$. For $k \in \mathbb{N}$, we let $U(k)$ denote the k–dimensional unitary group, and

$$\mathrm{Rep}_k(\Gamma) := \mathrm{Hom}(\Gamma, U(k))$$

the space[3] of k dimensional unitary representations of Γ. Define

$$\mathrm{Rep}(\Gamma) := \coprod_{k=1}^{\infty} \mathrm{Rep}_k(\Gamma).$$

Definition 3.1. Let X be a finite CW-complex, and let

$$\rho\colon\ X \to \mathrm{Rep}(\Gamma)$$

be a continuous map. We call ρ an *X-family of representations*, or simply a *family of representations*. We write ρ_x, a homomorphism from Γ to some $U(k)$, for the image of $x \in X$ under ρ.

[3] $\mathrm{Rep}_k(\Gamma)$ is given the subspace topology inherited from the product topology on $\mathrm{Map}(\Gamma, U(k)) = U(k)^\Gamma$.

Note that if $\rho : X \to \mathrm{Rep}(\Gamma)$ is a family of representations, then the restriction of ρ to any connected component of X must take values in $\mathrm{Rep}_k(\Gamma)$ for some fixed k.

Using an X-family as in the above definition, one may form a vector bundle over the space $B\Gamma \times X$ in the following way.

Definition 3.2. Let $\rho\colon X \to \mathrm{Rep}(\Gamma)$ be a family of representations. Write $X = X_1 \sqcup \cdots \sqcup X_n$ for the decomposition of X into connected components, and for each $i = 1, ..., n$ say the image of ρ restricted to X_i is contained in $\mathrm{Rep}_{k_i}(\Gamma)$.

Let $E\Gamma$ be the universal covering space of $B\Gamma$. Consider the space

$$\bigsqcup_{i=1}^{n} E\Gamma \times X_i \times \mathbb{C}^{k_i}$$

equipped with the Γ action defined by

$$g \cdot (z, x, v) := (gz, x, \rho_x(g)v).$$

The corresponding quotient space is a vector bundle over $B\Gamma \times X$, which we denote by E_ρ.

We denote by $[E_\rho] \in K^0(B\Gamma \times X) = K_0(C(X) \otimes C(B\Gamma))$ the topological K–theory class of this bundle. Abusing notation, we also write $[E_\rho]$ for the element $[E_\rho] \otimes 1_{\mathbb{Q}} \in K^0(B\Gamma \times X) \otimes \mathbb{Q}$.

Associated to each family of representations, we now obtain a "detecting map" as follows.

Definition 3.3. Let $\rho\colon X \to \mathrm{Rep}(\Gamma)$ and $[E_\rho] \in K^0(X \times B\Gamma)$ be as in Definitions 3.1 and 3.2 above. Then taking the slant product with $[E_\rho] \in K^0(X \times B\Gamma)$ defines a homomorphism

$$\begin{aligned} \rho_*\colon\ K_*(B\Gamma) &\to K^*(X) \\ x &\mapsto [E_\rho]/x. \end{aligned}$$

Abusing notation, we also write ρ_* for the homomorphism

$$\rho_* \otimes \mathrm{Id}_{\mathbb{Q}}\colon\ K_*(B\Gamma) \otimes \mathbb{Q} \to K^*(X) \otimes \mathbb{Q}$$

induced by ρ_*.

Definition 3.4. Let Γ be a (finitely presented) group with a finite model for the classifying space $B\Gamma$. A class x in the rational K–homology group $K_i(B\Gamma) \otimes \mathbb{Q}$ is said to be *flatly detectable* if there exists a family of representations

$$\rho\colon\ X \to \mathrm{Rep}(\Gamma)$$

such that

$$\rho_*(x) \in K^{-i}(X) \otimes \mathbb{Q}$$

is non-zero.

A group Γ is said to be in the class $\mathcal{FD}$ if it has a finite model for $B\Gamma$ and if all classes in $K_*(B\Gamma) \otimes \mathbb{Q}$ are flatly detectable.

The above terminology stems from the fact that when the parameter space X is a point and $B\Gamma$ is a smooth manifold, the bundle $E_\rho \to B\Gamma$ has a canonical flat connection.

Example 3.5. The trivial class $[1] \in K_0(B\Gamma)$ is always flatly detectable: one checks directly that it is detected by the trivial representation

$$\rho : \mathrm{pt} \to \mathrm{Hom}(\Gamma, U(1)).$$

The rest of this section is devoted to finding examples of groups in the class $\mathcal{FD}$. The first two results show that all finitely generated free groups are in $\mathcal{FD}$.

Proposition 3.6. *The group $\mathbb{Z}$ is in the class $\mathcal{FD}$.*

The proof we give below is based on an (unpublished) exposition of Higson-Roe of the proof of the Novikov conjecture for $\mathbb{Z}^n$ in Lusztig's thesis [17]. Corollary 3.17 also covers this case, but for the sake of variety, we give a different proof here.

Proof. We may of course take $B\mathbb{Z}$ to be a copy of the circle S^1, and will also take X to be a copy of S^1 (identified with the collection of complex numbers of modulus 1). Define $\rho\colon\ X \to \mathrm{Hom}(\mathbb{Z}, U(1))$ by

$$\rho_x\colon\ n \mapsto x^n.$$

Concretely, we may identify sections of the line bundle E_ρ over $S^1 \times S^1$ with the space of functions $f : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ that satisfy $f(z+n, x+m) = e^{inx} f(z,x)$ for all $n, m \in \mathbb{Z}$, $z \in \mathbb{R}$ and $x \in \mathbb{R}$. Now, the formulas

$$\nabla_z = \partial/\partial z \quad , \nabla_x = \partial/\partial x - 2\pi i z$$

define a connection on E_ρ, with curvature given by

$$R(\partial/\partial z,\ \partial/\partial x) = 2\pi i,$$

i.e. with curvature two-form given by $2\pi i dz \wedge dx$ (all of this is just direct computation). It follows from Chern-Weil theory that the Chern character of E_ρ is given by a generator of $H^2(S^1 \times S^1; \mathbb{R}) \cong \mathbb{R}$.

Now, by Example 3.5, it suffices to show that any non-zero element of $K_1(B\mathbb{Z}) \otimes \mathbb{Q} \cong \mathbb{Q}$ is flatly detectable. However, as is well-known, under the Chern isomorphism

$$Ch : K^0(S^1 \times S^1) \otimes \mathbb{Q} \cong H^{even}(S^1 \times S^1; \mathbb{Q}),$$

the element $2\pi i dz \wedge dx$ corresponds to the element $[u] \otimes [u] \in K^0(S^1 \times S^1) \otimes \mathbb{Q}$, where $u : S^1 \to U(1)$ is the canonical unitary identifying these spaces, which generates $K^1(S^1) \cong \mathbb{Z}$: up to rational multiples, which is all we need, this follows from the fact that the Chern character is a ring isomorphism, together with the Künneth formulas in cohomology and K–theory, and the fact that $Ch([u]) = [dx]$. The result follows from this, Lemma 2.4 part (2), and (rational) non-degeneracy of the pairing $K_1(S^1) \otimes K^1(S^1) \to \mathbb{Z}$. $\square$

Lemma 3.7. *Say Γ_1 and Γ_2 are groups in the class $\mathcal{FD}$. Then their free product $\Gamma = \Gamma_1 * \Gamma_2$ is in the class $\mathcal{FD}$.*

Proof. Let $B\Gamma_1$ and $B\Gamma_2$ be finite models for the classifying spaces of Γ_1 and Γ_2 respectively, and take $B\Gamma$ to be their wedge sum. Let x be a non-zero element of $K_*(B\Gamma) \otimes \mathbb{Q}$; we must show x is flatly detectable.

The Mayer-Vietoris sequence in K–homology implies that there is a natural decomposition

$$K_*(B\Gamma) \otimes \mathbb{Q} = (K_*(B\Gamma_1) \otimes \mathbb{Q}) \oplus (\widetilde{K}_*(B\Gamma_2) \otimes \mathbb{Q})$$

(and similarly with the roles of Γ_1 and Γ_2 reversed). Without loss of generality, assume that we can write $x = x_1 \oplus x_2$ with respect to this decomposition, where x_1 is non-zero. Using the assumption that Γ_1 is in the class $\mathcal{FD}$, there exists a family of representations $\rho^1 \colon X \to \mathrm{Rep}(\Gamma_1)$ such that $\rho^1_*(x_1) \neq 0$. The map ρ^1 gives rise to $\rho\colon X \to \mathrm{Rep}(\Gamma)$ by extending trivially on Γ_2 and using the universal property of the free product.

Now, the bundle E_ρ restricted to $B\Gamma_1 \times X \subseteq B\Gamma \times X$ is equal to E_{ρ^1} by construction, and is equal to an external product $\mathbb{C} \otimes F$ when restricted to $B\Gamma_2 \times X$, where $\mathbb{C}$ is the trivial bundle on $B\Gamma_2$ and F is some bundle over X (trivial on each connected component, but we do not need this). We then have that

$$\rho_*(x) = [E_\rho]/x = [E_{\rho^1}]/x_1 + (\mathbb{C} \otimes F)/x_2 = \rho^1_*(x_1) + \langle \mathbb{C}, x_2 \rangle \otimes F$$

using Lemma 2.4, part (2) and Example 2.3. As x_2 is an element of the reduced K–homology of $B\Gamma_2$, however, $\langle \mathbb{C}, x_2 \rangle = 0$, whence

$$\rho_*(x) = \rho^1_*(x_1) \neq 0$$

completing the proof. □

More generally, if Γ_1, Γ_2 are in the class $\mathcal{FD}$ and $\iota_i \colon A \to \Gamma_i$ are split inclusions for $i = 1, 2$, then the amalgamated free product $\Gamma = \Gamma_1 *_A \Gamma_2$ is in $\mathcal{FD}$; this follows from a minor elaboration of the argument above, which is omitted. It is not true that the class $\mathcal{FD}$ is preserved by arbitrary free products with amalgam: see Example 3.20 below.

Our next goal is to prove that the class $\mathcal{FD}$ is preserved under direct products, and thus in particular that it contains all finitely generated free abelian groups. In order to avoid using (somewhat non-trivial - [14, Chapter 9]) facts about external products in K–homology, the following definition is useful.

Definition 3.8. Let Γ be a (finitely presented) group with a finite model for the classifying space $B\Gamma$.

The Künneth theorem in K–theory (due to Atiyah - see [3, Corollary 2.7.15]) implies that the external product induces a natural isomorphism

$$(K^*(B\Gamma) \otimes \mathbb{Q}) \otimes_{\mathbb{Q}} (K^*(X) \otimes \mathbb{Q}) \cong K^*(B\Gamma \times X) \otimes \mathbb{Q}.$$

If $\phi\colon K^*(X)\otimes\mathbb{Q}\to\mathbb{Q}$ is any linear functional, we may thus define a natural map

$$K^*(B\Gamma\times X)\otimes\mathbb{Q}\xrightarrow{1\otimes\phi}(K^*(B\Gamma)\otimes\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{Q}\xrightarrow{\cong}K^*(B\Gamma)\otimes\mathbb{Q},$$

which by abuse of notation we denote $1\otimes\phi$. We write $K^*_{\mathcal{FD}}(B\Gamma)$ for the subset of $K^*(B\Gamma)\otimes\mathbb{Q}$ consisting of classes of the form $(1\otimes\phi)[E_\rho]$ where $\rho\colon X\to\mathrm{Rep}(\Gamma)$ is a family of representations, and $\phi\colon K^*(X)\otimes\mathbb{Q}\to\mathbb{Q}$ is a linear functional as above.

Lemma 3.9. *With notation as above, $K^*_{\mathcal{FD}}(B\Gamma)$ is a subspace of $K^*(B\Gamma)\otimes\mathbb{Q}$.*

Moreover, these two vector spaces are equal if and only if Γ is in the class $\mathcal{FD}$.

Proof. It is clear that $K^*_{\mathcal{FD}}(B\Gamma)$ is closed under scalar multiplication; we will show it is closed under addition. Let $(1\otimes\phi_i)[E_{\rho^i}]$ be elements of $K^*_{\mathcal{FD}}(B\Gamma)$ for $i=1,2$, where $\rho^i\colon X_i\to\mathrm{Rep}(\Gamma)$ and $\phi_i\colon K^*(X_i)\otimes\mathbb{Q}\to\mathbb{Q}$. Define

$$\rho\colon X_1\sqcup X_2\to\mathrm{Rep}(\Gamma)$$

by $\rho_x=\rho^i_x$ whenever $x\in X_i$, $i=1,2$. Then, after identifying $K^*(X_i)\otimes\mathbb{Q}$ with subspaces of $K^*(X_1\sqcup X_2)\otimes\mathbb{Q}$ in the natural way for $i=1,2$, we have $[E_\rho]=[E_{\rho^1}]+[E_{\rho^2}]$, and closure under addition follows from this.

The remaining claim follows from Lemma 2.4 part (2) and rational nondegeneracy of the pairing between K–theory and K–homology (see for example [14, Theorem 7.6.1]). □

Proposition 3.10. *Let Γ_1,Γ_2 be in the class $\mathcal{FD}$. Then the direct product $\Gamma=\Gamma_1\times\Gamma_2$ is in the class $\mathcal{FD}$.*

Proof. Let $B\Gamma_1$ and $B\Gamma_2$ be finite models for the classifying spaces of Γ_1 and Γ_2 respectively, and take $B\Gamma$ to be their direct product.

The Künneth theorem in K–theory [3, Corollary 2.7.15] implies that the external K–theory product induces a natural isomorphism

$$(K^*(B\Gamma_1)\otimes\mathbb{Q})\otimes(K^*(B\Gamma_2)\otimes\mathbb{Q})\cong K^*(B\Gamma_1\times B\Gamma_2)\otimes\mathbb{Q}\qquad(7)$$

of graded abelian groups. Identifying the two sides in line (7), Lemma 3.9 implies that it suffices to show that any class of the form $x_1\otimes x_2$, with $x_i\in K^*(B\Gamma_i)\otimes\mathbb{Q}$ for $i=1,2$, is in $K^*_{\mathcal{FD}}(B\Gamma)$.

Now, by assumption and Lemma 3.9, there exist families $\rho^i\colon X_i\to\mathrm{Rep}(\Gamma_i)$ and functionals $\phi_i\colon K^*(X_i)\otimes\mathbb{Q}\to\mathbb{Q}$ such that $x_i=(1\otimes\phi_i)[E_{\rho^i}]$. Define a new family $\rho\colon X_1\times X_2\to\mathrm{Rep}(\Gamma)$ by "pointwise tensor product"[4], i.e.

$$\rho\colon(x_1,x_2)\mapsto(\ (g_1,g_2)\mapsto\rho^1_{x_1}(g_1)\otimes\rho^2_{x_2}(g_2)\ ).$$

[4]This tensor product map arises from a choice of continuous tensor product map $U(n)\times U(m)\to U(nm)$; for example one may take the standard Kronecker product of matrices $(A,B)\mapsto A\otimes B$, which commutes with inverses, transposes, and conjugation, and hence maps $U(n)\times U(m)$ to $U(nm)$. Since the entries of $A\otimes B$ are just products of entries from A and B, continuity is immediate. Since $(A\otimes B)(C\otimes D)=AC\otimes BD$, this yields a continuous map $\mathrm{Hom}(\Gamma,U(n))\times\mathrm{Hom}(\Gamma,U(m))\to\mathrm{Hom}(\Gamma,U(nm))$.

From the construction of ρ, it follows that

$$[E_\rho] = [E_{\rho_1}] \otimes [E_{\rho_2}] \in K^0((X_1 \times B\Gamma_1) \times (X_2 \times B\Gamma_2))$$

and thus that (modulo Künneth isomorphisms)

$$1 \otimes \phi_1 \otimes \phi_2 \colon\ (K^*(B\Gamma) \otimes \mathbb{Q}) \otimes (K^*(X_1) \otimes \mathbb{Q}) \otimes (K^*(X_2) \otimes \mathbb{Q}) \to K^*(B\Gamma) \otimes \mathbb{Q}$$

takes $[E_\rho]$ to $x_1 \otimes x_2$ as required. □

Our next goal is to prove that $\mathcal{FD}$ passes to finite index supergroups; this combines with the previous results to imply, for example, that all torsion free crystallographic groups are in $\mathcal{FD}$. This requires an analysis of the transfer map in K–theory (see for example [2, Pages 250-1], where the image of a bundle under transfer is called the *direct image* bundle).

As an elementary treatment of the K–theory transfer seems to be missing from the literature, we give an essentially self-contained account below. See [26] for a treatment of transfer in a more general context. The treatment below is inspired by KK-theory, and could be developed completely in that context; we will not do this here.

Definition 3.11. Let Y be a finite CW complex with fundamental group Γ and universal cover $\widetilde{Y}$. Let Γ_0 be a finite index subgroup of Γ, and Y_0 the corresponding finite cover of Y.

Note that Y_0 is homeomorphic to the balanced product $\widetilde{Y} \times_\Gamma (\Gamma/\Gamma_0)$, whence $C(Y_0)$ is naturally isomorphic to

$$\mathcal{T}_Y^{Y_0} := C_b(\widetilde{Y} \times (\Gamma/\Gamma_0))^\Gamma = C_b(\widetilde{Y}, C(\Gamma/\Gamma_0))^\Gamma.$$

There is moreover clearly a right $C(Y) = C_b(\widetilde{Y})^\Gamma$ module-structure defined on $\mathcal{T}_Y^{Y_0}$.

The *transfer map*[5] in K–theory, denoted $t\colon\ K^*(Y_0) \to K^*(Y)$ is the homomorphism induced on finitely generated projective modules over $C(Y_0)$ by the formula

$$E \mapsto E \otimes_{C(Y_0)} \mathcal{T}_Y^{Y_0}.$$

The following simple lemma records some properties of the transfer map.

Lemma 3.12. *1. The K–theory transfer is well-defined.*

2. Let $\pi : Y_0 \to Y$ be a covering map and $t : K^*(Y_0) \to K^*(Y)$ be the corresponding transfer map as in Definition 3.11 above. Let E_{Γ_0} be the "flat"[6] bundle over Y induced by the quasi-regular representation of Γ on $l^2(\Gamma/\Gamma_0)$. Then the composition

$$t \circ \pi^*\colon\ K^*(Y) \to K^*(Y)$$

is equal to the (internal) K–theory product with $[E_{\Gamma_0}] \in K^*(Y)$.

In particular, $t \circ \pi^*$ is a rational isomorphism.

[5] It does of course agree with the more classical notion, as for example in [26, Pages 7-8], but we do not need this.

[6] Y may not be a manifold, so this does not literally make sense.

3. *With notation as in part (2), let $\rho : X \to Rep(\Gamma_0)$ be a family of representations. Let*
$$Ind(\rho) : X \to Rep(\Gamma)$$
be the family defined by "pointwise induction".[7] *Then if E_ρ, $E_{Ind(\rho)}$ are the bundles over $Y_0 \times X$ and $Y \times X$ defined by ρ, $Ind(\rho)$ respectively, we have that*
$$t[E_\rho] = [E_{Ind(\rho)}] \in K^*(Y).$$

4. *For any finite covering $Y_0 \to Y$ and any space X, the transfer map for the product covering $Y_0 \times X \to Y_0 \times X$ satisfies $t(y \times x) = t(y) \times x$ for all $y \in K^*(Y_0)$ and $x \in K^*(X)$.*

Proof. 1. For the case of K^0, this follows from the fact that $\mathcal{T}_Y^{Y_0}$ is finitely generated and projective, both as a left $C(Y_0)$ module and as a right $C(Y)$ module. The first of these is obvious - it is a free rank one module over $C(Y_0)$ - while the second follows from the fact that it is equal as a $C(Y)$ module to the sections of the bundle over Y induced by the representation of Γ on $l^2(\Gamma/\Gamma_0)$. The case of higher K–groups can be considered by taking suspensions (this is probably most easily seen with the "analysts suspension" – taking the tensor product with $C_0(\mathbb{R})$, and using K–theory with compact supports).

2. The homomorphism $\pi^* : C(Y) \to C(Y_0)$ induces a left $C(Y)$ module structure on $C(Y_0)$; write Π for the corresponding $C(Y)$–$C(Y_0)$ module. The composition $t \circ \pi^*$ is then equal to the map on $K^*(Y)$ induced by taking tensor product with the (finitely generated, projective) $C(Y)$ module
$$\Pi \otimes_{C(Y_0)} \mathcal{T}_Y^{Y_0};$$
this module simply *is* the sections of E_{Γ_0}, however.
The remaining statement follows from Chern-Weil theory in case Y and Y_0 are manifolds (not necessarily closed): indeed, rationally, taking the product with E_{Γ_0} is simply multiplication by $|\Gamma/\Gamma_0|$. The general case follows on replacing Y by a homotopy equivalent manifold (which need not be closed) and Y_0 with the corresponding cover.

3. Assume for simplicity of notation that X is connected (for the general case, consider each connected component separately), in which case we may assume that under each ρ_x, Γ acts on some fixed $\mathbb{C}^k$. The space of sections of E_ρ is then given by
$$C_b(\widetilde{Y} \times X, \mathbb{C}^k)^{\Gamma_0},$$
(where the fixed points are taken for a Γ_0 action analogous to the Γ action in Definition 3.2) while that for $E_{Ind(\rho)}$ is given by
$$C_b(\widetilde{Y} \times X, C_b(\Gamma, \mathbb{C}^k)^{\Gamma_0})^{\Gamma};$$

[7] Pointwise induction arises from a choice of continuous induction map $\mathrm{Hom}(\Gamma_0, U(n)) \to \mathrm{Hom}(\Gamma, U(n[\Gamma : \Gamma_0]))$. This map depends on a choice of coset representatives for Γ_0 in Γ. A detailed discussion can be found in [24]

it is not difficult to see that tensoring the former over $C(Y_0)$ by $\mathcal{T}_Y^{Y_0}$ yields the latter, which is the claim.

4. Let E, F be finitely generated projective modules over $C(Y_0)$, $C(X)$ respectively (i.e. spaces of sections of bundles over the respective spaces). It suffices to show that

$$(E \otimes F) \otimes_{C(X \times Y_0)} \mathcal{T}_{Y \times X}^{Y_0 \times X} \cong (E \otimes_{C(Y_0)} \mathcal{T}_Y^{Y_0}) \otimes F,$$

which is a straightforward computation.

$\square$

Lemma 3.13. *Say Γ is a group with finite classifying space, and that Γ_0 is a finite index subgroup of Γ in the class $\mathcal{FD}$. Then Γ is in the class $\mathcal{FD}$.*

Proof. Let $B\Gamma$ be a finite classifying space for Γ, and $B\Gamma_0$ the finite cover of $B\Gamma$ corresponding to the inclusion $\Gamma_0 \hookrightarrow \Gamma$, which is a finite classifying space for Γ_0. Let x be an element of $K^*(B\Gamma) \otimes \mathbb{Q}$; by Lemma 3.9, it suffices to show that x is in $K^*_{\mathcal{FD}}(B\Gamma)$. Now, by part (2) of Lemma 3.12,

$$t : K^*(B\Gamma_0) \otimes \mathbb{Q} \to K^*(B\Gamma) \otimes \mathbb{Q}$$

is surjective (where, as usual, we have abused notation, writing 't' for '$t \otimes Id_{\mathbb{Q}}$'), whence there exists $y \in K^*(B\Gamma_0) \otimes \mathbb{Q}$ with $t(y) = x$. As Γ_0 is in the class $\mathcal{FD}$, and by Lemma 3.9 again, there exist $\rho : X \to \mathrm{Rep}(\Gamma_0)$ and $\phi : K^*(X) \otimes \mathbb{Q} \to \mathbb{Q}$ such that $y = (1 \otimes \phi)[E_\rho]$. To complete the proof, note that the diagram

$$\begin{tikzcd}
K^*(B\Gamma_0 \times X) \otimes \mathbb{Q} \arrow[r, "t"] \arrow[d, "1\otimes\phi"] & K^*(B\Gamma \times X) \otimes \mathbb{Q} \arrow[d, "1\otimes\phi"] \\
K^*(B\Gamma_0) \otimes \mathbb{Q} \arrow[r, "t"] & K^*(B\Gamma) \otimes \mathbb{Q}
\end{tikzcd}$$

commutes by part (4) of Lemma 3.12, whence using part (3) of Lemma 3.12

$$x = t(y) = t((1 \otimes \phi)[E_\rho]) = (1 \otimes \phi)(t[E_\rho]) = (1 \otimes \phi)[E_{Ind(\rho)}]$$

and we are done. $\square$

Our next goal is to show that fundamental groups of compact, aspherical surfaces are in $\mathcal{FD}$. This will rely on Yang–Mills theory, and in particular on a result of the first author from [21]. To begin, we need to analyze the classifying map for the bundle E_ρ associated to a family of representations. In order to do this, we will need to consider a functorial model $\mathbb{B}(-)$ for classifying spaces, e.g. Milnor's infinite join construction or Segal's simplicial model. These have the property that homomorphisms $\rho\colon G \to H$ induce continuous maps $\mathbb{B}(\rho)\colon \mathbb{B}G \to \mathbb{B}H$, and in fact if G and H are topological groups, this gives rise to a continuous map

$$\mathrm{Hom}(G, H) \xrightarrow{\mathbb{B}} \mathrm{Map}_*(\mathbb{B}G, \mathbb{B}H).$$

(Continuity of this map is most easily checked using Segal's model, which gives a model for the classifying space so long as G and H are Lie groups, which suffices for our purposes.) A continuous map $\rho\colon X \to \mathrm{Hom}(G,H)$ now has an associated map $\mathbb{B}\circ\rho\colon X \to \mathrm{Map}_*(\mathbb{B}G,\mathbb{B}H)$, which has an adjoint $X\times\mathbb{B}G \to \mathbb{B}H$ (this adjoint is continuous so long as X is locally compact and Hausdorff, e.g. if X is a CW complex). We will denote this adjoint by $\rho^\vee$. The functorial model $\mathbb{B}(-)$ has an associated functorial model $\mathbb{E}(-)$ for the universal bundle, so that $\mathbb{E}G \to \mathbb{B}H$ is a universal (left) principal G–bundle. Moreover, there is a continuous mapping $\mathrm{Hom}(G,H) \xrightarrow{\mathbb{E}} \mathrm{Map}(\mathbb{E}G,\mathbb{E}H)$, such that for any $\rho\colon G\to H$, $\mathbb{E}(\rho)\colon \mathbb{E}G \to \mathbb{E}H$ is ρ–equivariant in the sense that $\mathbb{E}(\rho)(g\cdot e) = \rho(g)\cdot\mathbb{E}(\rho)(e)$. Moreover, the diagram

$$\begin{tikzcd} \mathbb{E}G \arrow[r, "\mathbb{E}(\rho)"] \arrow[d] & \mathbb{E}H \arrow[d] \\ \mathbb{B}G \arrow[r, "\mathbb{B}(\rho)"] & \mathbb{B}H \end{tikzcd} \tag{8}$$

commutes for each $\rho\colon G\to H$.

Lemma 3.14. *Let $\rho\colon X \to \mathrm{Hom}(\Gamma, U(n))$ be an X–family of representations. Let $B\Gamma$ be a finite model for the classifying space of Γ, and let $f\colon B\Gamma \to \mathbb{B}\Gamma$ be a classifying map for the universal Γ–bundle $E\Gamma \to B\Gamma$. Then the composite map*

$$B\Gamma\times X \xrightarrow{f\times Id_X} \mathbb{B}\Gamma\times X \xrightarrow{\rho^\vee} \mathbb{B}U(n)$$

is a classifying map for the principal $U(n)$–bundle associated to E_ρ.

Proof. The principal $U(n)$–bundle associated to E_ρ is simply

$$\begin{tikzcd} (E\Gamma\times X\times U(n))/\Gamma\ , \arrow[d] \\ B\Gamma\times X \end{tikzcd}$$

where Γ acts by $g\cdot(e,x,A) = (g\cdot e, x, \rho_x(g)A)$. This can be viewed as a *left* principal $U(n)$–bundle, via the action $A\cdot[e,x,B] = [e,x,BA^{-1}]$. There is then an analogous left principal $U(n)$–bundle over $\mathbb{B}\Gamma\times X$, formed by replacing $E\Gamma$ with $\mathbb{E}\Gamma$ in the previous construction. We will construct a commutative diagram of left principal $U(n)$–bundles as follows:

$$\begin{tikzcd} (E\Gamma\times X\times U(n))/\Gamma \arrow[r, "\tilde{f}\times\mathrm{Id}\times\mathrm{Id}"] \arrow[d] & (\mathbb{E}\Gamma\times X\times U(n))/\Gamma \arrow[r, "\alpha"] \arrow[d] & \mathbb{E}U(n) \arrow[d] \\ B\Gamma\times X \arrow[r, "f\times\mathrm{Id}"] & \mathbb{B}\Gamma\times X \arrow[r, "\rho^\vee"] & \mathbb{B}U(n) \end{tikzcd}$$

The map $\tilde{f}\colon E\Gamma \to \mathbb{E}\Gamma$ is the unique map of principal bundles covering f, and hence the left-hand square commutes by construction. Moreover, $\tilde{f}\times\mathrm{Id}\times\mathrm{Id}$

induces a $U(n)$–equivariant map between these quotient spaces, and hence the left-hand square is a pull-back diagram of principal $U(n)$–bundles. The map α is defined by

$$\alpha([e,x,A]) = A^{-1} \cdot \mathbb{E}(\rho_x)(e).$$

It follows from the properties of $\mathbb{E}(\rho_x)$ listed above that this is a $U(n)$–equivariant map, and that the right-hand diagram commutes. Thus the right-hand square is also a pullback diagram of principal $U(n)$–bundles, completing the proof. □

The next technical-but-simple lemma comes down to the relationship of the 'analysts suspension' $X \times \mathbb{R}$ and the 'topologists suspension' $X \wedge S^1$, and a way of making sense of the slant product on reduced K–theory and K–homology.

Lemma 3.15. *Let Γ be a group with a finite model $B\Gamma$ for its classifying space. Let x be an element of $\widetilde{K}_i(B\Gamma)$ which is non-zero after tensoring with $\mathbb{Q}$. Then for each $k \geq 0$ there exists*

$$y_k \in \widetilde{K}^i(B\Gamma \wedge S^{2k+i}) = \widetilde{K}^0(B\Gamma \wedge S^{2k})$$

such that if

$$\pi : B\Gamma \times S^{2k+i} \to B\Gamma \wedge S^{2k+i}$$

is the natural quotient map then the slant product $\pi^*(y_k)/x$ is a well-defined element of $K^*(S^{2k+i})$ and is non-zero.

Proof. Let $x_0 \in B\Gamma$ and $\infty \in S^{2k+i}$ be the respective basepoints, and identify $S^{2k+i}\backslash\{\infty\}$ with $\mathbb{R}^{2k+i}$. Recall that the K–theory (respectively, K–homology) of a locally compact, non-compact space Y is identified with the reduced K–theory (resp. K–homology) of the one point compactification Y^+, which is in turn identified with $K_*(C_0(Y))$ (resp. $K^*(C_0(Y))$). The statement of the lemma can thus be rewritten as follows: for any rationally non-trivial element $x \in K_i(C_0(B\Gamma\backslash\{x_0\}))$ and any $k \geq 0$ there exists

$$y_k \in K^i(C_0(B\Gamma\backslash\{x_0\}) \otimes C_0(\mathbb{R}^{2k+i}))$$

such that if

$$\iota : C_0(B\Gamma\backslash\{x_0\}) \otimes C_0(\mathbb{R}^{2k+i}) \to C_0(B\Gamma \times S^{2k+i}\backslash\{(x_0,\infty)\})$$

is the natural inclusion then

$$0 \neq \iota_*(y_k)/x \in K^0(S^{2k+i})$$

(here we think of $K_*(C_0(B\Gamma \times S^{2k+i}\backslash\{(x_0,\infty)\}))$ as a subring of $K_*(C(B\Gamma \times S^{2k+i}))$ to make sense of the slant product in the above).

This is not difficult, however: take any element $z \in K^i(C_0(B\Gamma))$ such that $x/z = \langle x, z\rangle$ is non-zero (which exists by rational non-degeneracy of the pairing), let $y_k = z \otimes b$, $b \in K^0(\mathbb{R}^{2k+i})$ the Bott generator, and apply Lemma 2.4 part 2. □

Theorem 3.16. *Let Γ be a group with a finite model $B\Gamma$ for its classifying space. Say there exists $K > 0$ such that for each $k > K$, there exists $N = N(k) > 0$ such that for $n > N$, the natural map*

$$\pi_k \mathrm{Hom}(\Gamma, U(n)) \xrightarrow{\mathbb{B}_*} \pi_k \mathrm{Map}_*(\mathbb{B}\Gamma, \mathbb{B}U(n))$$

is surjective. Then $\Gamma \in \mathcal{FD}$.

Proof. We need to show that each K–homology class on $B\Gamma$ is flatly detectable. Since the unit element is detected by the trivial representation, it will suffice to work with reduced K–homology. By Lemma 3.15, we know that for each rationally non-zero $x \in \widetilde{K}_i(B\Gamma)$ and each $k > 0$, there exists $y_k \in \widetilde{K}^0(B\Gamma \wedge S^{2k+i})$ such that $\pi^*(y_k)/x$ is non-zero, where π is the quotient map $B\Gamma \times S^{2k+i} \to B\Gamma \wedge S^{2k+i}$.

Choose k large enough that $2k+i > K$, and let $N = N(2k+i)$ be the number guaranteed by the hypothesis. Now y_k has the form $y_k = [V] - [W]$ for some bundles V, W over $B\Gamma \wedge S^{2k+i}$. The bundles V and W are then classified by maps $\alpha_V, \alpha_W \colon B\Gamma \wedge S^{2k+i} \to \mathbb{B}U(n)$ for some n, and we may assume that $n > N$. Moreover, we can assume these maps are based, and hence correspond to classes $\alpha_V, \alpha_W \in \pi_{2k+i}\mathrm{Map}_*(B\Gamma, \mathbb{B}U(n))$. Choose a classifying map $f \colon B\Gamma \to \mathbb{B}\Gamma$ for the universal bundle $E\Gamma \to B\Gamma$, and a homotopy inverse $g \colon \mathbb{B}\Gamma \to B\Gamma$ (note that g classifies $\mathbb{E}\Gamma$). By abuse of notation, g will also denote the induced map $\mathbb{B}\Gamma \wedge S^{2k+i} \to B\Gamma \wedge S^{2k+i}$ and the map $\mathrm{Map}_*(B\Gamma, \mathbb{B}U(n)) \to \mathrm{Map}_*(\mathbb{B}\Gamma, \mathbb{B}U(n))$ induced by pre-composition with g. Then g^*V and g^*W are classified by the adjoints of the elements $g_*\alpha_V$ and $g_*\alpha_W$ (respectively).

Our hypothesis now yields classes $\rho_V, \rho_W \in \pi_{2k+i}\mathrm{Hom}(\Gamma, U(n))$ such that $\mathbb{B}_*\rho_V = g_*\alpha_V$ and $\mathbb{B}_*\rho_W = g_*\alpha_W$. By Lemma 3.14 the bundle E_{ρ_V} is classified by the map

$$B\Gamma \times S^{2k+i} \xrightarrow{f \times \mathrm{Id}} \mathbb{B}\Gamma \times S^{2k+i} \xrightarrow{\rho_V^\vee} \mathbb{B}U(n).$$

By definition, $\rho_V^\vee = (b, z) = \alpha_V(z)(g(b))$. So in fact, E_{ρ_V} is classified by

$$B\Gamma \times S^{2k+i} \xrightarrow{f \times \mathrm{Id}} \mathbb{B}\Gamma \times S^{2k+i} \xrightarrow{g \times \mathrm{Id}} B\Gamma \times S^{2k+i} \xrightarrow{\pi} B\Gamma \wedge S^{2k+i} \xrightarrow{\alpha_V} \mathbb{B}U(n)$$

Since $g \circ f$ is homotopic to the identity, we conclude that $E_{\rho_V} \cong \pi^*V$. Similarly, we have $E_{\rho_W} \cong \pi^*W$.

Since $y_k = ([V] - [W])$, we have

$$0 \neq (\pi^* y_k)/x = ([\pi^*V] - [\pi^*W])/x = ([E_{\rho_V}] - [E_{\rho_W}])/x,$$

so we must have either $(\rho_V)_*(x) = [E_{\rho_V}]/x \neq 0$ or $(\rho_W)_*(x) = [E_{\rho_W}]/x \neq 0$; in either case we conclude that x is flatly detectable. $\square$

Corollary 3.17. *Let M^2 be a compact, aspherical surface (possibly with boundary). Then $\pi_1 M^2 \in \mathcal{FD}$.*

Proof. If $\partial M^2 \neq \emptyset$, then $\pi_1 M^2$ is isomorphic to a finitely generated free group F_m. This case follows from Proposition 3.6 and Lemma 3.7 above, but we give a different proof here for the sake of variety.

Note then that the natural map

$$\mathrm{Hom}(F_m, U(n)) = U(n)^m \longrightarrow \mathrm{Map}_*(\mathbb{B}F_m, \mathbb{B}U(n))$$

is a weak equivalence for each n: this map can be identified with the natural weak equivalence $U(n)^m \simeq (\Omega \mathbb{B}U(n))^m = \mathrm{Map}_*(\bigvee_m S^1, \mathbb{B}U(n)$, using the fact that $\mathbb{B}F_m \simeq \bigvee_m S^1$. For further details, see Ramras [21, Proof of Theorem 4.3]. Thus F_m satisfies the hypotheses of Theorem 3.16.

The case of closed, aspherical surfaces follows from Theorem 3.16 together with [21, Theorem 3.4], which states that the natural map

$$\mathrm{Hom}(\pi_1 M^2, U(n)) \longrightarrow \mathrm{Map}_*(\mathbb{B}\pi_1 M^2, \mathbb{B}U(n))$$

induces an isomorphism on homotopy groups in dimensions $0 < * < n-1$. □

For the readers' convenience, the next corollary summarizes what are perhaps the most natural examples we know to be in the class $\mathcal{FD}$.

Corollary 3.18. *The following classes of groups are in the class $\mathcal{FD}$:*

- *Finitely generated free groups.*
- *Finitely generated free abelian groups.*
- *Torsion free crystallographic groups.*
- *Fundamental groups of compact, aspherical surfaces.* □

On the other hand, the class $\mathcal{FD}$ seems likely to be quite restrictive. We suspect the following groups are not in the class $\mathcal{FD}$, although were unable to prove this (and would be very happy to be proved wrong!). The case of property (T) groups seems plausible as any family of finite dimensional representations (parametrized by a connected space) consists of representations that are all mutually equivalent, as is essentially contained in [8, Theorem 1.2.5]. The reason this does not yield a proof is that it does not preclude the existence of interesting topology *within* each equivalence class of representations.

Questions 3.19. Are the following groups in the class $\mathcal{FD}$:

- the integral Heisenberg group;
- (infinite) property (T) groups?

Here is an example that is definitely not in $\mathcal{FD}$.

Example 3.20. Recall the result due to Burger and Mozes [10] that there exist infinite simple groups of the form $\Gamma = F *_G F$, in which F and G are finitely generated free groups, and the amalgam is formed by including G as a finite index subgroup of F in two *different* ways. Moreover, the groups constructed in [10] have $B\Gamma$ a finite 2-dimensional CW complex. Such a group has *no* non-trivial finite-dimensional representations (due to the fact that linear groups are residually finite), whence the only flatly detectable classes in $K_*(B\Gamma)$ are multiples of the unit class. See also [27].

For these groups, $H_2(B\Gamma;\mathbb{Q})$ is rationally non-trivial: this follows from an elementary computation using the Mayer-Vietoris sequence for a free product with amalgam, and Euler characteristics of finite covers of graphs.

It follows that $\widetilde{K}_0(B\Gamma)\otimes\mathbb{Q}$ is non-trivial, and no class in it is flatly detectable. Note that the strong Novikov conjecture is certainly true for Γ as above, however, e.g. by using the results of [28].

Another source of similar examples is explained in Ramras [22, Section 2.1].

The questions below seem natural and interesting.

Questions 3.21.

- It follows from Example 3.20 that there exists free products with amalgam $\Gamma = \Gamma_1 *_A \Gamma_2$ such that Γ_1, Γ_2, A are all in $\mathcal{FD}$, but Γ is not. Are there 'reasonable' conditions on a free product with amalgam which imply it is in $\mathcal{FD}$?
- Are all torsion free one relator groups in $\mathcal{FD}$? If not, which ones are?
- Are all three-manifold groups in $\mathcal{FD}$?
- One could define a larger version of the class $\mathcal{FD}$ by considering families of *quasi-representations*, i.e. maps $\Gamma \to U(n)$ that agree with a homomorphism on a given set of generators up to some ϵ. What sort of groups have this weaker property (which should also imply the strong Novikov conjecture)? Recent work of Dadarlat [11, 12][8] investigating assembly and quasi-representations (among other things) seems very relevant here.

We end this section by noting some consequences of our results for the topology of unitary representation varieties. The identity map Id on $\mathrm{Hom}(\Gamma, U(n))$ may be viewed a the *universal* n–dimensional family of representations, and we denote the associated bundle by $\mathcal{U} = E_{\mathrm{Id}} \to \mathrm{Hom}(\Gamma, U(n)) \times B\Gamma$.

Proposition 3.22. *If* Γ *is in* $\mathcal{FD}$*, then for sufficiently large* n *there is a rationally injective map*

$$K_*(B\Gamma) \longrightarrow K^*(\mathrm{Hom}(\Gamma, U(n)))$$

given by $x \mapsto \mathcal{U}/x$*. Consequently, the sum of the (even, or odd) Betti numbers of* $\mathrm{Hom}(\Gamma, U(n))$ *is at least that of* Γ*.*

Proof. Since Γ is in $\mathcal{FD}$, every rational K–homology class $x \in K_*(B\Gamma)\otimes\mathbb{Q}$ satisfies $\rho_*(x) \neq 0 \in K^*(X)$ for some family of representations $\rho\colon X \to \mathrm{Hom}(\Gamma, U(n))$. By Part 1 of Lemma 2.4, we have $\rho^*([\mathcal{U}]/x) = [E_\rho]/x = \rho_*(x)$, so $[\mathcal{U}]/x \in K^*(\mathrm{Hom}(\Gamma, U(n)))\otimes\mathbb{Q}$ must be non-zero as well. As we have assumed $B\Gamma$ has the homotopy type of a finite CW complex, $K_*(B\Gamma)\otimes\mathbb{Q}$ is finitely generated, so any sufficiently large n works for all $x \in K_*(B\Gamma)$ (note here that $E_{\rho\oplus 1} = E_\rho \oplus E_1$, so $(\rho\oplus 1)_*(x) = \rho_*(x) + 1_*(x) = \rho_*(x)$ for any $x \in \widetilde{K}_*(B\Gamma)$). The statement about cohomology follows from consideration of the Chern character. □

[8] We would like to thank Marius Dadarlat for sharing these preprints with us.

4 Families of representations and analytic assembly

In this section, we relate groups in the class $\mathcal{FD}$ from Section 3 back to the analytic assembly map from Section 2. The main result is as follows.

Proposition 4.1. *For each family of representations $\rho\colon X \to Hom(\Gamma, U(k))$, the detecting map $\rho_*\colon K_*(B\Gamma) \to K^*(X)$ in Definition 3.3 factors through the analytic assembly map*

$$\mu\colon K_*(B\Gamma) \to K_*(C^*(\Gamma))$$

defined in Definition 2.7.

To prove this, we will need a lemma relating the Miscenko bundle M_Γ and the bundle E_ρ associated to ρ. Note first that if $\rho\colon X \to \mathrm{Rep}_k(\Gamma)$ is a family of representations, then ρ defines a $*$-homomorphism

$$\begin{aligned} \rho^\sharp\colon C^*(\Gamma) &\to M_k(C(X)) \\ u_g &\mapsto (\ x \mapsto \rho_x(g)\ ). \end{aligned}$$

Lemma 4.2. *The image of the Miscenko line bundle $[M_\Gamma] \in K_0(C^*(\Gamma) \otimes C(B\Gamma))$ under the map induced by the $*$-homomorphism*

$$\rho^\sharp \otimes 1_{C(B\Gamma)} : C^*(\Gamma) \otimes C(B\Gamma) \to M_k(C(X)) \otimes C(B\Gamma)$$

identifies naturally with the class $[E_\rho] \in K^0(B\Gamma \times X)$ from Definition 3.2 above.

Proof. Recall from the proof of Lemma 2.6 that the space of sections of the Miscenko bundle identifies naturally with $C_b(E\Gamma, C^*(\Gamma))^\Gamma$. It follows that the image of $[M_\Gamma]$ under $\rho^\sharp \otimes 1_{C(B\Gamma)}$ is the class in $K_*(C(B\Gamma \times X, M_k(\mathbb{C})))$ of the module

$$C_b(E\Gamma, C^*(\Gamma))^\Gamma \bigotimes_{C(B\Gamma, C^*(\Gamma))} C(B\Gamma \times X, M_k(\mathbb{C})),$$

where we have used the natural isomorphism $C(B\Gamma, M_k(C(X))) \cong C(B\Gamma \times X, M_k(\mathbb{C}))$, and the tensor product is defined via the left action of $C(B\Gamma, C^*(\Gamma))$ on $C(B\Gamma \times X, M_k(\mathbb{C}))$ coming from $\rho^\sharp \otimes 1_{C(B\Gamma)}$. Define now a Γ action on $C_b(E\Gamma \times X, M_k(\mathbb{C}))$ by

$$(g \cdot f)(z, x) := \rho_x(g) f(g^{-1}z, x),$$

and let $C_b(E\Gamma \times X, M_k(\mathbb{C}))^\Gamma$ denote the fixed points. There is an isomorphism of $C(B\Gamma \times X, M_k(\mathbb{C}))$ modules

$$C_b(E\Gamma, C^*(\Gamma))^\Gamma \bigotimes_{C(B\Gamma, C^*(\Gamma))} C(B\Gamma \times X, M_k(\mathbb{C})) \xrightarrow{\cong} C_b(E\Gamma \times X, M_k(\mathbb{C}))^\Gamma$$

defined for $f \in C_b(E\Gamma, C^*(\Gamma))^\Gamma$ and $h \in C(B\Gamma \times X, M_k(\mathbb{C}))$ by

$$f \otimes h \mapsto (\ (z,x) \mapsto \rho_x(f)h(z,x)\ )$$

(where we have extended ρ_x from Γ to $C^*(\Gamma)$). Hence the image of $[M_\Gamma]$ in $K_0(C(B\Gamma \times X, M_k(\mathbb{C})))$ is represented by the finitely generated projective module

$$C_b(E\Gamma \times X, M_k(\mathbb{C}))^\Gamma;$$

the essential point is that this is the space of sections of the endomorphism bundle of E_ρ, which completes the proof. More concretely, the image of this module under the Morita equivalence isomorphism

$$K_*(C(B\Gamma \times X, M_k(\mathbb{C}))) \cong K^*(B\Gamma \times X)$$

is given by

$$C_b(E\Gamma \times X, M_k(\mathbb{C}))^\Gamma \bigotimes_{C(B\Gamma \times X, M_k(\mathbb{C}))} C(B\Gamma \times X, \mathbb{C}^k).$$

Define a Γ action on $C_b(E\Gamma \times X, \mathbb{C}^k)$ by

$$(g \cdot f)(z,x) := \rho_x(g) f(g^{-1}z, x);$$

then there is an isomorphism

$$C_b(E\Gamma \times X, M_k(\mathbb{C}))^\Gamma \bigotimes_{C(B\Gamma \times X, M_k(\mathbb{C}))} C(B\Gamma \times X, \mathbb{C}^k) \xrightarrow{\cong} C_b(E\Gamma \times X, \mathbb{C}^k)^\Gamma,$$

defined for $f \in C_b(E\Gamma \times X, M_k(\mathbb{C}))^\Gamma$ and $h \in C(B\Gamma \times X, \mathbb{C}^k)$ by

$$f \otimes h \mapsto (\ (z,x) \mapsto f(z,x)h(\pi(z), x)\ )$$

(here $\pi \colon E\Gamma \to B\Gamma$ is the canonical quotient). However, $C_b(E\Gamma \times X, \mathbb{C}^k)^\Gamma$ is simply the space of sections of E_ρ, and we are done. □

Proof of Proposition 4.1. Using Definition 2.7, and Lemmas 2.4 and 4.2, we have that if $[M_\Gamma] \in K_0(C(B\Gamma) \otimes C^*(\Gamma))$ is the Miscenko bundle and $x \in K^*(B\Gamma)$, then

$$((\rho^\sharp)_* \circ \mu)(x) = (\rho^\sharp)_*([M_\Gamma]/x) = ((\rho^\sharp \otimes 1_{C(B\Gamma)})_*[M_\Gamma])/x = [E_\rho]/x = \rho_*(x),$$

where ρ_* is an is Definition 3.3. The result follows. □

The following corollary is an immediate consequence of Proposition 4.1.

Corollary 4.3. *Let Γ be group with finite classifying space $B\Gamma$ and let $x \in K_*(B\Gamma) \otimes \mathbb{Q}$ be a flatly detectable class. Then*

$$(\mu \otimes Id_{\mathbb{Q}})(x) \in K_*(C^*(\Gamma)) \otimes \mathbb{Q}$$

is non-zero.

In particular, if Γ is a group in the class $\mathcal{FD}$, then the analytic assembly map is rationally injective. □

This is enough to imply, for example, the Novikov conjecture for Γ (see [15]), and that (if a closed manifold) $B\Gamma$ does not admit a metric of positive scalar curvature (see [25]).

A The slant product and the Kasparov product

In the main part of the piece (Definition 2.2), we have introduced a slant product in operator K–theory in order to give an elementary picture of the assembly map. The more usual way of describing the assembly map is via the *Kasparov product*: see for example [16, Section 6], where the assembly map is denoted β. In this appendix, we show that our slant product agrees with the Kasparov product. More precisely, we prove the following result.

Proposition A.1. *The slant product*

$$K_i(A \otimes B) \otimes K^j(B) \to K_{i-j}(A)$$

from Definition 2.2 agrees naturally with the Kasparov product

$$KK_i(\mathbb{C}, A \otimes B) \otimes KK^j(B, \mathbb{C}) \to KK_{i-j}(\mathbb{C}, A).$$

Proof. For the sake of simplicity (and as it is the only case we need), assume that A and B are unital C^*-algebras. Using a suspension argument, it suffices to consider the case $i = j = 0$.

It suffices to show that if $p \in M_n(A \otimes B)$ defines a class $[p] \in K_0(A \otimes B)$ and $u \in \mathcal{D}(B)$ defines a class $[u] \in K^0(B)$, then (under the natural identifications of these groups with the corresponding KK-groups) the two products agree. We start by constructing KK-elements corresponding to p and u; we use the standard Kasparov picture of KK (with graded formalism).

- Let l^2 denote the Hilbert space $l^2(\mathbb{N})$. We may consider p as defining a bounded operator on the Hilbert-$(A \otimes B)$-module $A \otimes B \otimes l^2$, which is supported on
$$A \otimes B \otimes \text{span}\{\delta_0, ..., \delta_{n-1}\}.$$
Using Kasaprov's stabilization theorem [16, Page 151], there exists a partial isometry $v \in M(A \otimes B \otimes \mathcal{K}(l^2))$ such that $v^*v = 1 - p$ and $vv^* = 1$.

  Let $\widehat{l^2}$ denote the Hilbert space $l^2_{ev} \oplus l^2_{od}$, where each summand is a copy of l^2, and grade $\widehat{l^2}$ by stipulating that the first summand is even and the second odd. Define an operator F on
$$A \otimes B \otimes \widehat{l^2} \cong (A \otimes B \otimes l^2_{ev}) \oplus (A \otimes B \otimes l^2_{od}) \tag{9}$$
by
$$F = \begin{pmatrix} 0 & v^* \\ v & 0 \end{pmatrix}$$
(where of course the matrix decomposition reflects the direct sum decomposition in line (9)). The pair $(A \otimes B \otimes \widehat{l^2}, F)$ then defines an element

$[F] \in KK_0(\mathbb{C}, A \otimes B)$ which corresponds to $[p] \in K_0(A \otimes B)$ under the natural isomorphism $KK_0(\mathbb{C}, A \otimes B) \cong K_0(A \otimes B)$.

- Let $\mathcal{H}^B$ be the ample B-Hilbert space on which $\mathcal{D}(B)$ is defined, and let $\widehat{\mathcal{H}^B}$ denote the Hilbert space $\mathcal{H}^B_{ev} \oplus \mathcal{H}^B_{od}$, which is defined analogously to $\widehat{l^2}$ above. $\widehat{\mathcal{H}^B}$ is equipped with the natural action of B (by even operators). Define
$$G = \begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix} \in \mathcal{B}(\widehat{H}^B)$$
and note that the pair $(\widehat{\mathcal{H}^B}, G)$ defines an element $[G] \in KK^0(B, \mathbb{C})$ that corresponds to $[u] \in K^0(B)$ under the natural isomorphism $KK^0(B, \mathbb{C}) \cong K^0(B)$.

Note that to take the Kasparov product of $[F]$ and $[G]$, we must first replace $[G]$ with the element $[\widehat{\mathcal{H}^B} \otimes A, G \otimes 1] \in KK(A \otimes B, A)$; by an abuse of notation, however, we still denote this element $[G]$.

We must now compute the Kasparov product $[F] \otimes [G] \in KK_0(\mathbb{C}, A)$. The Hilbert-$A$-module for this element is

$$(A \otimes B \otimes \widehat{l^2}) \bigotimes_{A \otimes B} (\widehat{\mathcal{H}^B} \otimes A) \cong A \otimes \widehat{l^2} \otimes \widehat{\mathcal{H}^B}. \tag{10}$$

We will use the decomposition

$$(A \otimes l^2_{ev} \otimes \mathcal{H}^B_{ev}) \oplus (A \otimes l^2_{od} \otimes \mathcal{H}^B_{od}) \oplus (A \otimes l^2_{ev} \otimes \mathcal{H}^B_{od}) \oplus (A \otimes l^2_{od} \otimes \mathcal{H}^B_{ev})$$

of this module to write operators on it as 4×4 matrices. Note now that a $G \otimes 1$-connection (see [9, Section 18.3]) is given by

$$\hat{G} = \begin{pmatrix} 0 & 0 & 1 \otimes u^* & 0 \\ 0 & 0 & 0 & -1 \otimes u^* \\ 1 \otimes u & 0 & 0 & 0 \\ 0 & -1 \otimes u & 0 & 0 \end{pmatrix},$$

whence [9, Proposition 18.10.1] implies that the product $[F] \otimes [G]$ can be represented by the pair

$$(A \otimes \widehat{l^2} \otimes \widehat{\mathcal{H}^B}\ ,\ F \hat{\otimes} 1 + ((1 - F^2)^{\frac{1}{2}} \hat{\otimes} 1)\hat{G}). \tag{11}$$

Now, the natural (even) action of $A \otimes B \otimes \mathcal{K}(l^2)$ on $A \otimes \mathcal{H}^B \otimes l^2$ extends to the multiplier algebra, so we may treat the operators p, v and v^* as acting directly on $A \otimes \mathcal{H}^B \otimes l^2$. Having adopted this convention, the operator from line (11) above is equal to

$$\begin{pmatrix}0 & v^*\\ v & 0\end{pmatrix}\hat{\otimes}1+\Big(\begin{pmatrix}1-v^*v & 0\\ 0 & 1-vv^*\end{pmatrix}^{\frac{1}{2}}\hat{\otimes}1\Big)\begin{pmatrix}0 & 0 & 1\otimes u^* & 0\\ 0 & 0 & 0 & -1\otimes u^*\\ 1\otimes u & 0 & 0 & 0\\ 0 & -1\otimes u & 0 & 0\end{pmatrix}$$

$$=\begin{pmatrix}0 & 0 & 0 & v^*\\ 0 & 0 & v & 0\\ 0 & v^* & 0 & 0\\ v & 0 & 0 & 0\end{pmatrix}+\begin{pmatrix}p & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & p & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 1\otimes u^* & 0\\ 0 & 0 & 0 & -1\otimes u^*\\ 1\otimes u & 0 & 0 & 0\\ 0 & -1\otimes u & 0 & 0\end{pmatrix}$$

$$=\begin{pmatrix}0 & 0 & p(1\otimes u^*) & v^*\\ 0 & 0 & v & 0\\ p(1\otimes u) & v^* & 0 & 0\\ v & 0 & 0 & 0\end{pmatrix}.$$

Passing back to the ungraded picture, the class of the cycle

$$\Big(A\otimes\widehat{l^2}\otimes\widehat{\mathcal{H}^B}\ ,\ \begin{pmatrix}0 & 0 & p(1\otimes u^*) & v^*\\ 0 & 0 & v & 0\\ p(1\otimes u) & v^* & 0 & 0\\ v & 0 & 0 & 0\end{pmatrix}\Big)$$

in $KK_0(\mathbb{C},A)$ corresponds under the isomorphism

$$KK_0(\mathbb{C},A)\cong K_1\Big(\frac{M(A\otimes\mathcal{K})}{A\otimes\mathcal{K}}\Big)\quad(\ \cong K_0(A)\ )$$

to the K_1-class defined by

$$\begin{pmatrix}p(1\otimes u) & v^*\\ v & 0\end{pmatrix}\in M(A\otimes\mathcal{K})$$

(this element is indeed unitary modulo $A\otimes\mathcal{K}$). Modulo $A\otimes\mathcal{K}$, however, we have that

$$\begin{pmatrix}p(1\otimes u) & v^*\\ v & 0\end{pmatrix}=\begin{pmatrix}p(1\otimes u)p & v^*\\ v & 0\end{pmatrix}=\begin{pmatrix}p(1\otimes u)p+(1-p) & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}p & v^*\\ v & 0\end{pmatrix};$$

moreover, the second matrix in the product satisfies $X^2=I$, and is thus K–theoretically trivial. Hence the class we have is

$$\Big[\begin{pmatrix}p(1\otimes u)p+(1-p) & 0\\ 0 & 1\end{pmatrix}\Big]\in K_1\Big(\frac{M(A\otimes\mathcal{K})}{A\otimes\mathcal{K}}\Big);$$

using the fact that the inclusion

$$\frac{A\otimes\mathcal{B}}{A\otimes\mathcal{K}}\hookrightarrow\frac{M(A\otimes\mathcal{K})}{A\otimes\mathcal{K}}$$

induces an isomorphism on K–theory and the argument of Example 2.3, however, it is not difficult to see that the image of this in $K_0(A)$ is precisely the same as the slant product $[p]/[u]$ from Definition 2.2. □

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