An index theorem for band-dominated operators with slowly oscillating coefficients (after Deundyak and Shteinberg)

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Abstract. We provide a proof of an index theorem for band-dominated operators with slowly oscillating coefficients. The statement is essentially the same as the main result of the announcement [5] of Deundyak and Shteinberg, but our methods are very different from those hinted at there. The index theorem we prove can also be seen as a partial generalization to higher dimensions of the main result of the article [14] of Rabinovich, Roch and Roe.

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1. Introduction

In this piece we prove an index theorem for band-dominated operators (BDOs) with slowly oscillating coefficients on $l^2(\mathbb{Z}^N)$; this is a special case of the results of Deundyak and Shteinberg announced in [5]. It also partially generalizes results of Rabinovich, Roch and Roe [14] (see [16] for a different proof), who give an index theorem for BDOs on $l^2(\mathbb{Z})$.

The main ingredient in our argument is K-theoretic, making use of the stable Higson corona of Emerson and Meyer [8], and asymptotic morphisms in the sense of Connes and Higson [10] to facilitate certain computations. We have given relatively elementary demonstrations of some properties of the stable Higson corona elsewhere [23].

The other important ingredient is an index theorem of Semenjuta and Simonenko [20] for so-called *generalized discrete convolution operators* (these are called BDOs with *continuous coefficients* in [15]).[1] Indeed, our approach

[1] An earlier version of this paper used the Atiyah-Singer index theorem for pseudodifferential operators on the n-torus [2] instead of [20]; we would like to thank the referee for pointing out the current more direct approach.

is to use the K-theory computations alluded to above to reduce the general case to the index theorem of [20]; note, however, that the final statement makes no use of K-theory – see Theorem 6.1 below.

We have not been able to prove an index theorem for general BDOs (i.e. with coefficients that are not necessarily slowly oscillating) on $l^2(\mathbb{Z}^N)$; indeed Corollary 6.2 and Remark 6.3 below imply the existence of a 'dimension obstruction' to straightforwardly extending the current result to the general case (see also [15, pages 151-2]). The result below is also restricted to the case of BDOs on Hilbert space, due to our reliance on C^*-algebraic methods; note, however, the results of Roch [17] imply that the main theorems also apply to *band* operators with slowly oscillating coefficients on l^p-spaces.

Outline of the piece

Section 2 elaborates on [19, Section 4] to give a picture of the symbol calculus for BDOs in terms of C^*-algebra crossed products. This is used extensively in the following computations, and may be of some interest in its own right – see [24, Section 2] for some developments along these lines. Section 3 introduces BDOs with slowly oscillating coefficients, following [15, Section 2.4]. Section 4 introduces BDOs with continuous coefficients, and states the index theorem of Semenjuta and Simonenko [20] that applies in this case. Section 5 introduces the stable Higson corona and gives a K-theoretic statement and proof of the main result. Finally, Section 6 gives a non-K-theoretic restatement of the main theorem, as well as pointing out the 'dimension obstruction' alluded to above, and sketching an index theorem for *locally compact* operators along the lines of that of Rabinovich and Roch in [13].

Notation

Throughout the piece, $m = (m_1, ..., m_N)$ is used to denote an N-tuple of integers in $\mathbb{Z}^N$. $l^2(\mathbb{Z}^N)$ denotes the Hilbert space of complex-valued square summable functions on $\mathbb{Z}^N$; we will denote its usual basis by $\{\delta_m : m \in \mathbb{Z}^N\}$. $\mathcal{L}(E)$ denotes the algebra of bounded operators on a Banach space E, and $\mathcal{K}(E)$ the compact operators. For k a positive integer, $M_k(\mathbb{C})$ denotes the algebra of $k \times k$ matrices over $\mathbb{C}$. If X is a compact Hausdorff topological space, $C_k(X)$ denotes the C^*-algebra of continuous functions from X to $M_k(\mathbb{C})$ (equipped with pointwise operations and supremum norm); if X is just assumed locally compact then $l^\infty_k(X)$ denotes all bounded functions from X to $M_k(\mathbb{C})$ and $C_{0,k}(X)$ denotes the continuous functions from X to $M_k(\mathbb{C})$ that vanish at infinity. The symbol '$\cdot \otimes \cdot$' always denotes a completed tensor product: either of Hilbert spaces, or the spatial tensor product of C^*-algebras. Finally, $K_*(A) := K_0(A) \oplus K_1(A)$ denotes the topological K-theory group of a C^*-algebra A, and $K^*(X)$ the topological K-theory group of a space X.

2. Crossed products of C^*-algebras and Fredholm theory for BDOs

Definition 2.1. Let $l^\infty_k(\mathbb{Z}^N)$ act on $l^2(\mathbb{Z}^N) \otimes \mathbb{C}^k$ by pointwise matrix multiplication; throughout the piece, we will abuse notation by writing f for both a function $f : \mathbb{Z}^N \to M_k(\mathbb{C})$ in $l^\infty_k(\mathbb{Z}^N)$ and the corresponding operator in $\mathcal{L}(l^2(\mathbb{Z}^N) \otimes \mathbb{C}^k)$. For $m \in \mathbb{Z}^N$, let $V_m \in \mathcal{L}(l^2(\mathbb{Z}^N) \otimes \mathbb{C}^k)$ be the left unitary shift operator defined by

$$V_m : \delta_n \otimes v \mapsto \delta_{m+n} \otimes v$$

for any $n \in \mathbb{Z}^N$ and $v \in \mathbb{C}^k$. A *band-dominated operator (BDO)* is an element of the C^*-subalgebra of $\mathcal{L}(l^2(\mathbb{Z}^N) \otimes \mathbb{C}^k)$ generated by $l^\infty_k(\mathbb{Z}^N)$ and the shifts $\{V_m : m \in \mathbb{Z}^N\}$; denote this C^*-algebra by $\mathcal{A}_k$.

Any BDO[2] $T \in \mathcal{A}_k$ has a unique formal representation as a (possibly infinite) sum

$$T = \sum_{m \in \mathbb{Z}^N} f_m V_m$$

where $f_m \in l^\infty_k(\mathbb{Z}^N)$ for all $m \in \mathbb{Z}^N$ (the sum need not converge, even in the weak operator topology, however). We call the elements f_m appearing in such a representation the *coefficients* of T.

The following proposition is essentially due to Higson and Yu. It generalizes [14, Proposition 2.1]; the proof is a slight adaptation of that of [4, Proposition 5.1.3] and is thus omitted.

Proposition 2.2. *Let α be the natural left shift action of $\mathbb{Z}^N$ on $l^\infty_k(\mathbb{Z}^N)$ (which is spatially implemented by the unitaries V_m). Let B be a C^*-subalgebra of $l^\infty(\mathbb{Z}^N)$ that is preserved by α and let $B_k := B \otimes M_k(\mathbb{C})$ be concretely represented on $l^2(\mathbb{Z}^N) \otimes \mathbb{C}^k$ by pointwise matrix multiplication.*

Then the C^*-subalgebra of $\mathcal{L}(l^2(\mathbb{Z}^N) \otimes \mathbb{C}^k)$ generated by elements of the form bV_m, $b \in B_k$, $m \in \mathbb{Z}^N$, is canonically isomorphic to the reduced crossed product $B_k \rtimes_r \mathbb{Z}^N$ of B_k by $\mathbb{Z}^N$ with respect to the action α.

Assume moreover that B_k contains $C_0(\mathbb{Z}^N) \otimes M_k(\mathbb{C}) \cong C_{0,k}(\mathbb{Z}^N)$. Then the isomorphism above takes the subalgebra $C_{0,k}(\mathbb{Z}^N) \rtimes_r \mathbb{Z}^N$ of $B_k \rtimes_r \mathbb{Z}^N$ to $\mathcal{K}(l^2(\mathbb{Z}^N) \otimes \mathbb{C}^k)$. □

Corollary 2.3. *$\mathcal{A}_k$ is canonically isomorphic to $l^\infty_k(\mathbb{Z}^N) \rtimes_r \mathbb{Z}^N$.*

Proof. Set B in the above to be all of $l^\infty(\mathbb{Z}^N)$. □

Let B be a unital C^*-subalgebra of $l^\infty(\mathbb{Z}^N)$ which contains $C_0(\mathbb{Z}^N)$ and is preserved under the shift action α of $\mathbb{Z}^N$. The Gelfand-Naimark theorem implies that B is canonically isomorphic to $C(\overline{\mathbb{Z}^N})$ for some compact Hausdorff space $\overline{\mathbb{Z}^N}$ which is an *equivariant compactification* of $\mathbb{Z}^N$: $\overline{\mathbb{Z}^N}$ contains $\mathbb{Z}^N$ as an open dense subset in such a way that the (left) shift action of $\mathbb{Z}^N$ on itself extends to a continuous action of $\mathbb{Z}^N$ on $\overline{\mathbb{Z}^N}$ by homeomorphisms.

[2]Indeed, any bounded operator on $l^2(\mathbb{Z}^N) \otimes \mathbb{C}^k$.

Corollary 2.4. *Let* $\overline{\mathbb{Z}^N}$ *be an equivariant compactification of* $\mathbb{Z}^N$ *and* $\partial\mathbb{Z}^N := \overline{\mathbb{Z}^N}\backslash\mathbb{Z}^N$ *the associated* corona *space. Then there exists a short exact sequence of* C^**-algebras*

$$0 \to \mathcal{K}(l^2(\mathbb{Z}^N)\otimes\mathbb{C}^k) \to C_k(\overline{\mathbb{Z}^N}) \rtimes_r \mathbb{Z}^N \xrightarrow{\sigma} C_k(\partial\mathbb{Z}^N) \rtimes_r \mathbb{Z}^N \to 0.$$

Proof. There is a short exact sequence of C^*-algebras

$$0 \to C_{0,k}(\mathbb{Z}^N) \to C_k(\overline{\mathbb{Z}^N}) \to C_k(\partial\mathbb{Z}^N) \to 0,$$

where all the $*$-homomorphisms are equivariant for the natural (left) $\mathbb{Z}^N$ actions on each algebra. As $\mathbb{Z}^N$ is an exact group in the sense of C^*-algebra theory (see for example [4, Chapter 5]), this gives rise to a short exact sequence of crossed product algebras

$$0 \to C_{0,k}(\mathbb{Z}^N) \rtimes_r \mathbb{Z}^N \to C_k(\overline{\mathbb{Z}^N}) \rtimes_r \mathbb{Z}^N \to C_k(\partial\mathbb{Z}^N) \rtimes_r \mathbb{Z}^N \to 0.$$

Proposition 2.2 implies that $C_k(\overline{\mathbb{Z}^N}) \rtimes_r \mathbb{Z}^N$ is naturally represented on $l^2(\mathbb{Z}^N)\otimes\mathbb{C}^k$ as the C^*-algebra generated by $C_k(\overline{\mathbb{Z}^N})$ and the shifts V_m, and moreover that this representation takes $C_{0,k}(\mathbb{Z}^N) \rtimes_r \mathbb{Z}^N$ to the compact operators on $l^2(\mathbb{Z}^N)\otimes\mathbb{C}^k$, so we are done. □

As a corollary, note that an operator

$$F \in C_k(\overline{\mathbb{Z}^N}) \rtimes_r \mathbb{Z}^N \subseteq \mathcal{L}(l^2(\mathbb{Z}^N)\otimes\mathbb{C}^k)$$

is Fredholm if and only if its 'symbol' $\sigma(F) \in C_k(\partial\mathbb{Z}^N) \rtimes_r \mathbb{Z}^N$ is invertible. This is closely related to the results on Fredholmness of BDOs in [15], in particular the symbol calculus from [15, Section 2.2.4]; we will not need this in this piece, but see [24, Section 2] for a study of the precise relationship and some corollaries.

3. The Higson compactification and BDOs with slowly oscillating coefficients

In this section we introduce BDOs with slowly oscillating coefficients.

Definition 3.1. Let X be a locally compact metric space, and let $f : X \to E$ be a continuous bounded function from X to a normed space E. f is called *slowly oscillating* if for all $R > 0$ the function $\nabla_R f$ defined by

$$(\nabla_R f)(x) = \sup\{\|f(x) - f(y)\| : d(x,y) \leq R\}$$

tends to zero at infinity in X.

The previous definition restricts to the class of functions studied in [15, Section 2.4.1] in the case $X = \mathbb{Z}^N$ (equipped with the restriction of the Euclidean metric from $\mathbb{R}^N$), but we need it in slightly more generality.

Definition 3.2. The slowly oscillating functions from X to $\mathbb{C}$ form a commutative C^*-subalgebra when equipped with the supremum norm, which we denote $SO(X)$; we write SO for $SO(\mathbb{Z}^N)$ when there is no risk of confusion.

The C^*-algebra of slowly oscillating functions on X with values in $M_k(\mathbb{C})$ is denoted $SO_k(X)$ or just SO_k when $X = \mathbb{Z}^N$.

The space of multiplicative linear functionals on $SO(X)$ is denoted $\overline{X}^h$, and the associated corona $\partial_h X := \overline{X}^h \backslash X$ (here X is identified with the space of point evaluations on $SO(X)$). We call $\overline{X}^h$ the *Higson compactification* of X and $\partial_h X$ its *Higson corona*; see Remark 3.3 below for a justification of the name.

Finally, the C^*-subalgebra of $\mathcal{L}(l^2(\mathbb{Z}^N) \otimes \mathbb{C}^k)$ generated by SO_k (acting by pointwise matrix multiplication) and the shifts V_m will be denoted $\mathcal{A}(SO_k)$ or $\mathcal{A}(SO_k(\mathbb{Z}^N))$ if we need to be more specific . Operators in this algebra are called *BDOs with slowly oscillating coefficients.*

Remark 3.3. $\overline{X}^h$ has been extensively studied in coarse geometry, index theory and K-homology; the name 'Higson compactification' is after Nigel Higson, who introduced it in these areas [11]. The only examples we will use in this piece are the Higson compactifications of $X = \mathbb{Z}^N$ and $X = \mathbb{R}^N$. Note that in [15, Section 2.4], the authors use the notation $M(SO)$ and $M^\infty(SO)$ for what we have called $\overline{\mathbb{Z}^N}^h$ and $\partial_h \mathbb{Z}^N$ respectively.

Now, $SO(\mathbb{Z}^N) \subseteq l^\infty(\mathbb{Z}^N)$ is unital, contains $C_0(\mathbb{Z}^N)$, and is preserved by the shift action α of $\mathbb{Z}^N$ on $l^\infty(\mathbb{Z}^N)$. Hence $\overline{\mathbb{Z}^N}^h$ is an equivariant compactification of $\mathbb{Z}^N$ in the sense of the previous section, whence Corollary 2.4 gives a short exact sequence

$$0 \to \mathcal{K}(l^2(\mathbb{Z}^N) \otimes \mathbb{C}^k) \to \mathcal{A}_k(SO) \xrightarrow{\sigma} C_k(\partial_h \mathbb{Z}^N) \rtimes_r \mathbb{Z}^N \to 0. \tag{1}$$

Lemma 3.4. *The action of $\mathbb{Z}^N$ on $\partial_h \mathbb{Z}^N$ is trivial.*

Proof. It suffices to prove that if f is an element of SO_k, η is a functional in $\partial_h \mathbb{Z}^N$ and $m \in \mathbb{Z}^N$, then $\eta(V_m^* f V_m - f) = 0$. Let $(m^i)_{i\in I}$ be any net in $\mathbb{Z}^N$ converging to η in the Gelfand topology on $M(SO_k)$, so in particular (m^i) tends to infinity in $\mathbb{Z}^N$. It follows that

$$\begin{aligned} \eta(V_m^* f V_m - f) &= \lim_{i\in I} \big((V_m^* f V_m)(m^i) - f(m^i) \big) \\ &= \lim_{i\in I} \big(f(m^i - m) - f(m^i) \big) = 0 \end{aligned}$$

by definition of slowly oscillating functions. □

Hence one has the identifications

$$\begin{aligned} C_k(\partial_h \mathbb{Z}^N) \rtimes_r \mathbb{Z}^N &\cong C_k(\partial_h \mathbb{Z}^N) \otimes C_r^*(\mathbb{Z}^N) \cong C_k(\partial_h \mathbb{Z}^N) \otimes C(\mathbb{T}^N) \\ &\cong C_k(\partial_h \mathbb{Z}^N \times \mathbb{T}^N), \end{aligned}$$

where the second isomorphism uses the Fourier transform to identify the reduced group C^*-algebra of $\mathbb{Z}^N$ and the continuous functions on the N-torus $\mathbb{T}^N$. The short exact sequence in line (1) above can thus be rewritten

$$0 \to \mathcal{K}(l^2(\mathbb{Z}^N) \otimes \mathbb{C}^k) \to \mathcal{A}_k(SO) \xrightarrow{\sigma} C_k(\partial_h \mathbb{Z}^N \times \mathbb{T}^N) \to 0. \tag{2}$$

We will not need this fact, but it is not hard to check that σ as in the above line is the same as the map smb from [15, page 102].

4. The spherical compactification and BDOs with continuous coefficients

In this section we introduce the *spherical* compactification of $\mathbb{Z}^N$. The spherical compactification is used to define *BDOs with continuous coefficients* in [15, Section 2.3.6]; we repeat the definition below for the reader's convenience. We then state an index theorem of Semenjuta and Simonenko [20] for BDOs with continuous coefficients; this is an important ingredient in our main result.

Definition 4.1. Let S^{N-1} be the $N-1$ dimensional unit sphere considered as a subset of N-dimensional Euclidean space $\mathbb{R}^N$. Let $\overline{\mathbb{Z}^N}^s$ be equal as a set to the disjoint union $\mathbb{Z}^N \sqcup S^{N-1}$, and topologize it by stipulating that:

- the induced subspace topology on S^{N-1} is the usual one;
- a sequence $(m^k)_{k=0}^{\infty}$ in $\mathbb{Z}^N\backslash\{0\}$ converges to $x \in S^{N-1}$ if and only if the sequence of norms $(\|m^k\|_{\mathbb{R}^N})$ converges to infinity, and the sequence $(m^k/\|m^k\|_{\mathbb{R}^N})$ converges to $x \in S^{N-1}$.

Equipped with this (metrizable) topology, $\overline{\mathbb{Z}^N}^s$ will be called the *spherical compactification* of $\mathbb{Z}^N$.

The natural action of $\mathbb{Z}^N$ on itself extends to an action on the spherical compactification; in the language of Section 2, this says that $\overline{\mathbb{Z}^N}^s$ is an equivariant compactification of $\mathbb{Z}^N$. In particular, Corollary 2.4 above implies that there is a short exact sequence of C^*-algebras

$$0 \to \mathcal{K}(l^2(\mathbb{Z}^N) \otimes \mathbb{C}^k) \to C_k(\overline{\mathbb{Z}^N}^s) \rtimes_r \mathbb{Z}^N \to C_k(S^{N-1}) \rtimes_r \mathbb{Z}^N \to 0. \tag{3}$$

However, it follows from the argument of Lemma 3.4 that the action of $\mathbb{Z}^N$ is trivial on the sphere at infinity, whence

$$C_k(S^{N-1})\rtimes_r\mathbb{Z}^N \cong C_k(S^{N-1})\otimes C_r^*(\mathbb{Z}^N) \cong C_k(S^{N-1}\times\mathbb{T}^N) \cong C_k(S^{N-1}\times\mathbb{T}^N),$$

just as in the case of $C(\partial_h\mathbb{Z}^N) \rtimes_r \mathbb{Z}^N$ looked at earlier. The short exact sequence in line (3) above thus becomes

$$0 \to \mathcal{K}(l^2(\mathbb{Z}^N) \otimes \mathbb{C}^k) \to C_k(\overline{\mathbb{Z}^N}^s) \rtimes_r \mathbb{Z}^N \stackrel{\sigma_S}{\to} C_k(S^{N-1} \times \mathbb{T}^N) \to 0. \tag{4}$$

The map σ_S is called the *Semenjuta-Simonenko symbol map* in what follows. In [15, Section 2.3.6], BDOs in the subalgebra $C(\overline{\mathbb{Z}^N}^s) \rtimes_r \mathbb{Z}^N$ of $\mathcal{A}_k$ are called *BDOs with continuous coefficients.*

In [21][3] Semenjuta and Simonenko prove an index formula for operators in $C_k(\overline{\mathbb{Z}^N}^s) \rtimes_r \mathbb{Z}^N$, which we now describe.

[3]See also [20] for the definitions used in [21].

Assume that $k = N$ in the discussion above, and that $F \in C_N(\overline{\mathbb{Z}^N}^s) \rtimes_r \mathbb{Z}^N$ is Fredholm, so that $\sigma_S(F)$ is a continuous map

$$\sigma_S(F) : S^{N-1} \times \mathbb{T}^N \to GL_N(\mathbb{C});$$

from $S^{N-1} \times \mathbb{T}^N$ into the N-dimensional complex general linear group. Let now $GL_{N-1}(\mathbb{C})$ be embedded in $GL_N(\mathbb{C})$ as the stabilizer of the point $(1, 0, ..., 0) \in \mathbb{C}^N$, so that $GL_N(\mathbb{C})/GL_{N-1}(\mathbb{C})$ identifies with $\mathbb{C}^N \backslash \{0\}$; we may deform retract this latter space onto the sphere of radius 1 in $\mathbb{C}^N$, a copy of S^{2N-1}. This process gives rise to a map

$$\widetilde{\sigma_S(F)} : S^{N-1} \times \mathbb{T}^N \to GL_N(\mathbb{C}) \to \mathbb{C}^N \backslash \{0\} \to S^{2N-1}. \tag{5}$$

Fix orientations on the domain and codomain of $\widetilde{\sigma_S(F)}$ as follows: that on S^{2N-1} is the orientation it inherits from the complex structure on $\mathbb{C}^N$; that on $S^{N-1} \times \mathbb{T}^N$ is the product orientation where S^{N-1} (respectively, $\mathbb{T}^N$) inherits its orientation as a subspace (respectively, quotient) of the same copy of $\mathbb{R}^N$ (how the original $\mathbb{R}^N$ is orientated makes no difference). Now, as $\widetilde{\sigma_S(F)}$ is a continuous map between oriented manifolds of the same dimension, it has an integer degree $Degree(\widetilde{\sigma_S(F)})$.

The following theorem is a restatement of [20, formula (2), page 135], the main theorem of that paper. This theorem can also be seen as a special case of index formulas of Fedosov [9] and Atiyah-Singer [2].[4]

Theorem 4.2. *Say F, $\widetilde{\sigma_S(F)}$ are as above. Then*

$$Index(F) = (-1)^{\frac{N(N+1)}{2}-1} \frac{Degree(\widetilde{\sigma_S(F)})}{(N-1)!}.$$

Note that a map $S^{N-1} \times \mathbb{T}^N \to GL_k(\mathbb{C})$ for *any* $k \in \mathbb{N}$ can be considered as a map $S^{N-1} \times \mathbb{T}^N \to GL_{k'}(\mathbb{C})$ where $k' = \max\{k, N\}$ and $GL_k(\mathbb{C})$ is either identified with $GL_{k'}(\mathbb{C})$, or embedded in it as a subgroup in the natural way. From here, the map can be homotoped (through maps with range in $GL_{k'}(\mathbb{C})$) to one of the form

$$S^{N-1} \times \mathbb{T}^N \to GL_N(\mathbb{C}) \oplus \{1_{k'-N}\} \subseteq GL_{k'}(\mathbb{C}),$$

where $k' = \max\{k, N\}$ and $1_{k'-N}$ is the identity in $k' - N$ dimensions (see [1, page 239]). Thus one may define a map

$$Ind_S : K^1(S^{N-1} \times \mathbb{T}^N) \to \mathbb{Z} \tag{6}$$

by first homotoping a class $[x]$ so that it has image in $GL_N(\mathbb{C}) \oplus \{1_{k'-N}\}$, and then applying the formula from Theorem 4.2 to the part of the image in $GL_N(\mathbb{C})$; it is easy to see that this is well-defined on the level of K-theory. Note that if $[\sigma_S(F)] \in K^1(S^{N-1} \times \mathbb{T}^N)$ is the class defined by the symbol of a Fredholm operator F, then

$$Ind_S([\sigma_S(F)]) = Index(F). \tag{7}$$

[4] We thank the referee for pointing out the references [20] and [9].

We will use this later. The homomorphism in line (6) will be called the *Semenjuta-Simonenko index map*; note, of course, that it is just a concrete instantiation of the usual index map in K-theory arising from the short exact sequence in line (4) – indeed, this follows from the formula in line (7).

5. The stable Higson compactification and proof of the main result

In this section we give a proof of the main index theorem (Theorem 5.4 below, restated later as Theorem 6.1).

The main ingredient in the proof is the stable Higson corona of Emerson and Meyer [8], which is used to 'organize' certain K-theoretic computations. The stable Higson corona is not a topological corona associated to some compactification of $\mathbb{Z}^N$, but rather a noncommutative C^*-algebra that plays a similar role: the point is that the Higson corona $\partial_h\mathbb{Z}^N$ is well-known to have very bad topological properties [12], [6], [7]; we use the *stable* Higson corona as a manageable substitute for it on a K-theoretical level.

The following definition comes from [8].

Definition 5.1. Let $\mathcal{K}$ be an abstract copy of the C^*-algebra of compact operators on a separable infinite dimensional Hilbert space and X be a locally compact metric space. The *stable Higson compactification* of X, denoted $\bar{\mathfrak{c}}(X)$, is the C^*-algebra of (continuous, bounded) slowly oscillating functions from X to $\mathcal{K}$.

Note that it contains the C^*-algebra of continuous functions from X to $\mathcal{K}$ that vanish at infinity, denoted $C_0(X,\mathcal{K})$, as an ideal. The *stable Higson corona* of X, denoted $\mathfrak{c}(X)$, is the quotient C^*-algebra

$$\mathfrak{c}(X) := \frac{\bar{\mathfrak{c}}(X)}{C_0(X,\mathcal{K})}.$$

The following proposition collects together some basic facts from coarse geometry that will be of use to us here; all are simple special cases of more general facts.

Proposition 5.2. 1. *There is a natural inclusion $C_k(\overline{\mathbb{Z}^N}^s) \hookrightarrow C_k(\overline{\mathbb{Z}^N}^h)$, which fixes $C_{0,k}(\mathbb{Z}^N)$ and thus passes to an inclusion of quotients*

$$C_k(S^{N-1}) \cong \frac{C_k(\overline{\mathbb{Z}^N}^s)}{C_{0,k}(\mathbb{Z}^N)} \hookrightarrow \frac{C_k(\overline{\mathbb{Z}^N}^h)}{C_{0,k}(\mathbb{Z}^N)} \cong C_k(\partial_h\mathbb{Z}^N).$$

2. *There is an inclusion $C_k(\overline{\mathbb{Z}^N}^h) \hookrightarrow \bar{\mathfrak{c}}(\mathbb{Z}^N)$, which maps $C_{0,k}(\mathbb{Z}^N)$ into $C_0(\mathbb{Z}^N,\mathcal{K})$ and thus passes to an inclusion of quotients*

$$C_k(\partial_h\mathbb{Z}^N) \cong \frac{C_k(\overline{\mathbb{Z}^N}^h)}{C_{0,k}(\mathbb{Z}^N)} \hookrightarrow \frac{\bar{\mathfrak{c}}(\mathbb{Z}^N)}{C_0(\mathbb{Z}^N,\mathcal{K})} \cong \mathfrak{c}(\mathbb{Z}^N).$$

3. *The inclusion of* $\mathbb{Z}^N$ *into* $\mathbb{R}^N$ *induces* $*$*-homomorphisms*
$$C_k(\overline{\mathbb{R}^N}^h) \to C_k(\overline{\mathbb{Z}^N}^h) \quad and \quad \bar{\mathfrak{c}}(\mathbb{R}^N) \to \bar{\mathfrak{c}}(\mathbb{Z}^N);$$
these induce isomorphisms on the 'corona algebras'
$$C_k(\partial_h\mathbb{R}^N) \cong C_k(\partial_h\mathbb{Z}^N) \quad and\ \mathfrak{c}(\mathbb{R}^N) \cong \mathfrak{c}(\mathbb{Z}^N)$$
respectively.

Sketch of proofs. For part (1), note that the restriction of any function in $C(\overline{\mathbb{Z}^N}^s)$ to $\mathbb{Z}^N$ is slowly oscillating when restricted to $\mathbb{Z}^N$, hence extends (uniquely) to $\overline{\mathbb{Z}^N}^h$; this defines the inclusions in the statement.

For part (2), identify $M_k(\mathbb{C})$ with a sub-C^*-algebra of $\mathcal{K}$ via any $*$-homomorphism taking rank one projections to rank one projections. The restriction of a function in $C_k(\overline{\mathbb{Z}^N}^h)$ to $\mathbb{Z}^N$ is then a slowly oscillating function from $\mathbb{Z}^N$ to $\mathcal{K}$; this defines the inclusions in the statement. Note that while this map involves a choice of embedding $M_k(\mathbb{C}) \hookrightarrow \mathcal{K}$, the choice does not matter on the level of K-theory.

For part (3), the $*$-homomorphisms are both defined by restriction of functions from $\mathbb{R}^N$ to $\mathbb{Z}^N$. The isomorphisms in the statement come from the fact that the inclusion $\mathbb{Z}^N \hookrightarrow \mathbb{R}^N$ is a *coarse equivalence*, and that the Higson corona and stable Higson coronas define functors on the *coarse category*; for details, see [18, Proposition 2.41] for the case of the Higson corona and [8, Proposition 13] for the stable Higson corona. □

Consider now the map $C(\overline{\mathbb{R}^N}^h) \to C_b([1,\infty), C(S^{N-1}))$ defined by sending a function f on the Higson compactification of $\mathbb{R}^N$ to the map sending $t \in [1,\infty)$ to the restriction of f to the sphere of radius t about zero in $\mathbb{R}^N$. Passing to quotients by functions vanishing at infinity and using the second part of proposition 5.2 above, this gives rise to a map
$$\alpha : C(\partial_h\mathbb{Z}^N) \cong \frac{C(\overline{\mathbb{R}^N}^h)}{C_0(\mathbb{R}^N)} \to \frac{C_b([1,\infty), C(S^{N-1}))}{C_0([1,\infty), C(S^{N-1}))}.$$
In particular, then, α is an *asymptotic morphism* from $C(\partial_h\mathbb{Z}^N)$ to $C(S^{N-1})$ in the sense of [10], so induces a map on K-theory
$$\alpha_* : K^*(\partial_h\mathbb{Z}^N) \to K^*(S^{N-1}). \tag{8}$$
Analogously, there is a $*$-homomorphism
$$\alpha^{\mathfrak{c}} : \mathfrak{c}(\mathbb{Z}^N) \cong \frac{\bar{\mathfrak{c}}(\mathbb{Z}^N)}{C_0(\mathbb{Z}^N, \mathcal{K})} \to \frac{C_b([1,\infty), C(S^{N-1}) \otimes \mathcal{K})}{C_0([1,\infty), C(S^{N-1}) \otimes \mathcal{K})}$$
i.e. an asymptotic morphism from $\mathfrak{c}(\mathbb{Z}^N)$ to $C(S^{N-1}) \otimes \mathcal{K}$. It thus induces a map on K-theory
$$\alpha^{\mathfrak{c}}_* : K_*(\mathfrak{c}(\mathbb{Z}^N)) \to K_*(C(S^{N-1}) \otimes \mathcal{K}). \tag{9}$$

Remarks 5.3.

- The asymptotic morphism α acts on K-theory roughly as follows; see for example [3, Chapter 25] for details. Assume that $[p] \in K^0(\partial_h \mathbb{Z}^N)$, where p is a projection in $M_k(C(\partial_h \mathbb{Z}^N))$. Let $\tilde{p}$ be any lift of p to $M_k(C(\overline{\mathbb{R}^N}^h))$; this exists because of the canonical identification $C(\partial_h \mathbb{Z}^N) \cong C(\partial_h \mathbb{R}^N)$. Let $\alpha_t(p)$ be the restriction of $\tilde{p}$ to the sphere of radius t in $\mathbb{R}^N$ for all $t > 0$. One then has that $\alpha_t(p)^2 - \alpha_t(p)$ and $\alpha_t(p)^* - \alpha_t(p)$ both tend to zero as $t \to \infty$, i.e. $\alpha_t(p)$ gets 'close' to being a projection as $t \to \infty$. It thus (via the functional calculus) defines a class in $K^0(S^{N-1})$ for all t suitably large; we let $\alpha_*[p]$ be the class of any $\alpha_t(p)$ for t large. The action of α on K^1, and that of $\alpha^{\mathfrak{c}}$ on K-theory, are described similarly.
- In the notation of Deundyak and Shteinberg [5], α_* is equal to the composition $i_1^* \circ (i_\infty^*)^{-1}$ as a map on K-theory; in this sense, one can think of α_* as a concrete way of describing the map $i_1^* \circ (i_\infty^*)^{-1}$. We do not need this fact here, so will not prove it, but see [22, Chapters 5, 6].

We may now state the main result of this section.

Theorem 5.4. *Let $F \in \mathcal{A}(SO_k(\mathbb{Z}^N))$ be a Fredholm BDO on $\mathbb{Z}^N$ with slowly oscillating coefficients. Its symbol $\sigma(F) \in C_k(\partial_h \mathbb{Z}^N \times \mathbb{T}^N)$ as in line* (2) *above is thus invertible and so defines a class $[\sigma(F)] \in K^1(\partial_h \mathbb{Z}^N \times \mathbb{T}^N)$.*

Let $Ind_S : K^1(S^{N-1} \times \mathbb{T}^N) \to \mathbb{Z}$ be the Semenjuta-Simonenko index map of line (6) *above. Then*

$$Index(F) = Ind_S \circ (\alpha \otimes 1)_*[\sigma(F)].$$

Theorem 5.4 is proved by studying the commutative diagram introduced in line (10) below. The vertical arrows between the bottom and middle rows in this diagram are the inclusions from the first part of Proposition 5.2, while those between the second and third rows are as in the second part of that proposition.

$$\begin{tikzcd}
0 \arrow[r] & C_0(\mathbb{Z}^N, \mathcal{K}) \arrow[r] & \overline{\mathfrak{c}}(\mathbb{Z}^N) \arrow[r] & \mathfrak{c}(\mathbb{Z}^N) \arrow[r] & 0 . \\
0 \arrow[r] & C_{0,k}(\mathbb{Z}^N) \arrow[r] \arrow[u] & C_k(\overline{\mathbb{Z}^N}^h) \arrow[r] \arrow[u] & C_k(\partial_h \mathbb{Z}^N) \arrow[r] \arrow[u, "j^h"] & 0 \\
0 \arrow[r] & C_{0,k}(\mathbb{Z}^N) \arrow[r] \arrow[u, equal] & C_k(\overline{\mathbb{Z}^N}^s) \arrow[r] \arrow[u] & C_k(S^{N-1}) \arrow[r] \arrow[u, "j^s"] & 0
\end{tikzcd} \tag{10}$$

The maps in this diagram are all $\mathbb{Z}^N$-equivariant. We may thus take crossed products by $\mathbb{Z}^N$ everywhere, getting a new commutative diagram.

$$
\begin{tikzcd}
0 \arrow[r] & (C_0(\mathbb{Z}^N) \rtimes_r \mathbb{Z}^N) \otimes \mathcal{K} \arrow[r] & \bar{\mathfrak{c}}(\mathbb{Z}^N) \rtimes_r \mathbb{Z}^N \arrow[r] & \mathfrak{c}(\mathbb{Z}^N) \rtimes_r \mathbb{Z}^N \arrow[r] & 0 . \\
0 \arrow[r] & C_{0,k}(\mathbb{Z}^N) \rtimes_r \mathbb{Z}^N \arrow[r] \arrow[u] & C_k(\overline{\mathbb{Z}^N}^h) \rtimes_r \mathbb{Z}^N \arrow[r] \arrow[u] & C_k(\partial_h \mathbb{Z}^N) \rtimes_r \mathbb{Z}^N \arrow[r] \arrow[u, "j^h \rtimes_r 1"'] & 0 \\
0 \arrow[r] & C_{0,k}(\mathbb{Z}^N) \rtimes_r \mathbb{Z}^N \arrow[r] \arrow[u, equal] & C_k(\overline{\mathbb{Z}^N}^s) \rtimes_r \mathbb{Z}^N \arrow[r] \arrow[u] & C_k(S^{N-1}) \rtimes_r \mathbb{Z}^N \arrow[r] \arrow[u, "j^s \rtimes_r 1"'] & 0
\end{tikzcd}
\tag{11}
$$

Note that, as $\mathbb{Z}^N$ is an exact group, the rows are still exact.

We make the following identifications in the above:

- all the $\mathbb{Z}^N$ actions in the right-hand column are trivial (by the argument of Lemma 3.4), whence '$\cdot \rtimes_r \mathbb{Z}^N$' has the same effect as '$\cdot \otimes C(\mathbb{T}^N)$';
- by Proposition 2.2, $C_k(\overline{\mathbb{Z}^N}^h) \rtimes_r \mathbb{Z}^N \cong \mathcal{A}(SO_k(\mathbb{Z}^N))$ and $C_{0,k}(\mathbb{Z}^N) \rtimes_r \mathbb{Z}^N \cong \mathcal{K}(l^2(\mathbb{Z}^N) \otimes \mathbb{C}^k) \cong \mathcal{K}(l^2(\mathbb{Z}^N)) \otimes M_k(\mathbb{C})$.

Diagram (11) above thus becomes

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{K}(l^2(\mathbb{Z}^N)) \otimes \mathcal{K} \arrow[r] & \bar{\mathfrak{c}}(\mathbb{Z}^N) \rtimes_r \mathbb{Z}^N \arrow[r] & \mathfrak{c}(\mathbb{Z}^N) \otimes C(\mathbb{T}^N) \arrow[r] & 0 , \\
0 \arrow[r] & \mathcal{K}(l^2(\mathbb{Z}^N)) \otimes M_k(\mathbb{C}) \arrow[r] \arrow[u] & \mathcal{A}(SO_k(\mathbb{Z}^N)) \arrow[r] \arrow[u] & C_k(\partial_h \mathbb{Z}^N \times \mathbb{T}^N) \arrow[r] \arrow[u, "j^h \otimes 1"'] & 0 \\
0 \arrow[r] & \mathcal{K}(l^2(\mathbb{Z}^N)) \otimes M_k(\mathbb{C}) \arrow[r] \arrow[u, equal] & C_k(\overline{\mathbb{Z}^N}^s) \rtimes_r \mathbb{Z}^N \arrow[r] \arrow[u] & C_k(S^{N-1} \times \mathbb{T}^N) \arrow[r] \arrow[u, "j^s \otimes 1"'] & 0
\end{tikzcd}
\tag{12}
$$

which in turn gives rise to a commutative diagram of six-term exact sequences in K-theory. We will be interested only in the 'index map' portion of these exact sequences, as studied below.

Note that the map

$$
\mathcal{K}(l^2(\mathbb{Z}^N)) \otimes M_k(\mathbb{C}) \to \mathcal{K}(l^2(\mathbb{Z}^N)) \otimes \mathcal{K}
$$

between the middle and top rows on the left-hand-side of (12) above induces an isomorphism on K-theory. Hence the portion of the commutative diagram of six term exact sequences containing the index maps looks like

$$
\begin{tikzcd}
K_1(\mathfrak{c}(\mathbb{Z}^N) \otimes C(\mathbb{T}^N)) \arrow[r, "Ind_{\mathfrak{c}}"] & \mathbb{Z} . \\
K^1(\partial_h \mathbb{Z}^N \times \mathbb{T}^N) \arrow[r, "Ind"] \arrow[u, "(j^h \otimes 1)_*"] & \mathbb{Z} \arrow[u, equal] \\
K^1(S^{N-1} \times \mathbb{T}^N) \arrow[r, "Ind_S"] \arrow[u, "(j^s \otimes 1)_*"] & \mathbb{Z} \arrow[u, equal]
\end{tikzcd}
\tag{13}
$$

Here the horizontal maps are all connecting maps ('index maps') in the K-theory six term exact sequences associated to (12); the bottom one is of course simply the Semenjuta-Simonenko index map from line (6) in Section 4 above.

To complete the proof of Theorem 5.4, we need the following two lemmas.

Lemma 5.5. *Let α_*, $\alpha_*^{\mathfrak{c}}$ and j^h be as in lines* (8), (9) *and* (10) *above respectively. They are related by the formula*

$$\alpha_* = \alpha_*^{\mathfrak{c}} \circ j_*^h : K^*(\partial_h \mathbb{Z}^N) \to K^*(S^{N-1}),$$

where we have identified $C_k(\partial_h \mathbb{Z}^N)$ and $C(\partial_h \mathbb{Z}^N, \mathcal{K})$ on the level of K-theory in order to make sense of this.

Proof. Both maps are given by the same formulas up to the natural isomorphisms on K-theory induced by the inclusions $C_k(\partial_h \mathbb{Z}^N) \hookrightarrow C(\partial_h \mathbb{Z}^N, \mathcal{K})$ and $C_k(S^{N-1}) \hookrightarrow C(S^{N-1}, \mathcal{K})$ □

Lemma 5.6. *$j^h \circ j^s : C_k(S^{N-1}) \to \mathfrak{c}(\mathbb{Z}^N)$ induces an isomorphism on K-theory, with inverse $\alpha_*^{\mathfrak{c}}$.*

Moreover, there are maps between $K_*(C_k(S^{N-1} \times \mathbb{T}^N))$ and $K_*(\mathfrak{c}(\mathbb{Z}^N) \otimes C(\mathbb{T}^N))$ functorially induced by $j^h \circ j^s$ and α_*^S which also induce mutually inverse isomorphisms on K-theory.

Proof. As $\mathbb{R}^N$ is a $CAT(0)$ space, up to the isomorphisms in K-theory induced by the inclusion $M_k(\mathbb{C}) \hookrightarrow \mathcal{K}$ and the canonical identification of $\mathfrak{c}(\mathbb{Z}^N)$ and $\mathfrak{c}(\mathbb{R}^N)$, the map $j^h \circ j^s$ is the same as the map i from [23, Proposition 4.2][5], which is an isomorphism.

Consider the map induced on K-theory by $\alpha^{\mathfrak{c}} \circ j^h \circ j^s$. Pre-composing with the stabilization isomorphism again, this takes a class $[f]$ from $K_*(C(S^{N-1}, \mathcal{K}))$[6] to the restriction of any lift $\tilde{f}$ of f to the sphere of radius t for t suitably large (cf. Remark 5.3). However, as f is just a function on S^{N-1}, we may assume that $\tilde{f}$ simply *is* f on all spheres of radius t for $t \geq 1$. Hence $\alpha^{\mathfrak{c}} \circ j^h \circ j^s$ induces the identity on K-theory.

This says that $\alpha_*^{\mathfrak{c}}$ is a one-sided inverse to $j_*^h \circ j_*^s$; as the latter map is an isomorphism, however, $\alpha_*^{\mathfrak{c}}$ must be a two-sided inverse as required.

To complete the proof, it suffices to show that the $*$-homomorphism

$$(j^h \circ j^s) \otimes 1 : C(S^{N-1}) \otimes \mathcal{K} \otimes C(\mathbb{T}^N) \to \mathfrak{c}(\mathbb{Z}^N) \otimes C(\mathbb{T}^N)$$

and asymptotic morphism

$$\alpha^{\mathfrak{c}} \otimes 1 : \mathfrak{c}(\mathbb{Z}^N) \otimes C(\mathbb{T}^N) \to \frac{C_b([1,\infty), C(S^{N-1}) \otimes \mathcal{K} \otimes C(\mathbb{T}^N))}{C_0([1,\infty), C(S^{N-1}) \otimes \mathcal{K} \otimes C(\mathbb{T}^N))}$$

[5] This also follows from results in [8].

[6] Stability of $C(S^{N-1}, \mathcal{K})$ implies that classes of this form for $f \in C(S^{N-1}, \mathcal{K})$ generate this K-theory group.

induce mutually inverse isomorphisms at the level of K-theory. A slight adaptation of [23, Proposition 4.3] again shows that $((j^h \circ j^s) \otimes 1)_*$ is an isomorphism, however, and then the same argument as above shows that $(\alpha^{\mathfrak{c}} \otimes 1)_*$ is a one-sided, whence also two-sided, inverse. □

We are now finally ready to prove Theorem 5.4.

Proof of Theorem 5.4. Let F be as in Theorem 5.4. It suffices to compute $Ind[\sigma(F)]$, which is equal to

$$Ind_{\mathfrak{c}} \circ (j^h \otimes 1)_*[\sigma(F)] = Ind_{\mathfrak{c}} \circ (j^h \otimes 1)_* \circ (j^s \otimes 1)_* \circ (\alpha^{\mathfrak{c}} \otimes 1)_* \circ (j^h \otimes 1)_*[\sigma(F)]$$

using naturality of the index map in K-theory and Lemma 5.6. Using naturality of the index map again, this is in turn equal to

$$Ind_S \circ (\alpha^{\mathfrak{c}} \otimes 1)_* \circ (j^h \otimes 1)_*[\sigma(F)].$$

Finally, applying Lemma 5.5, this is equal to

$$Ind_S \circ (\alpha \otimes 1)_*[\sigma(F)],$$

which proves Theorem 5.4. □

6. Restatement of the main result

In this section we give a more concrete restatement of Theorem 5.4, which is much closer to the statement of [5]. We also discuss some ‘dimension corollaries’, and sketch an extension of Theorem 5.4 along the lines of [13].

Theorem 6.1. *Let $F \in \mathcal{A}(SO_N(\mathbb{Z}^N))$ be a Fredholm BDO on $\mathbb{Z}^N$ with slowly oscillating coefficients and values in $M_N(\mathbb{C})$. Then its index may be computed by the following ‘recipe’:*

- *Using the fact that $\partial_h \mathbb{Z}^N = \partial_h \mathbb{R}^N$, extend all the matrix entries of*

$$(f_{ij}) := \sigma(F) : \partial_h \mathbb{Z}^N \times \mathbb{T}^N \to \mathbb{C}$$

  to functions on $\mathbb{R}^N \times \mathbb{T}^N$.[7] Denote by $\widetilde{f_{ij}}$ the extension of f_{ij} so obtained.
- *For some $t_0 \in [1, \infty)$ and all $t \geq t_0$, the matrix $(\widetilde{f_{ij}})$ will be invertible when restricted to the sphere of radius t. It thus defines a function, which we denote $\sigma_t(F)$, from $S^{N-1} \times \mathbb{T}^N$ to $GL_N(\mathbb{C})$.*
- *For any $t \geq t_0$, there is thus a map $\widetilde{\sigma_t(F)} : S^{N-1} \times \mathbb{T}^N \to S^{2N-1}$ defined as in line (5) above. Just as in Theorem 4.2, then*

$$Index(F) = (-1)^{\frac{N(N+1)}{2}-1} \frac{Degree(\widetilde{\sigma_t(F)})}{(N-1)!}.$$

  for any $t \geq t_0$. □

[7]This sounds difficult. In concrete situations, however, the coefficients of the original F will be given as restrictions of slowly oscillating functions on $\mathbb{R}^N$, so it is not really a problem.

Just as in Section 4, if $F \in \mathcal{A}(SO_k(\mathbb{Z}^N))$ for some $k \in \mathbb{N}$, then for t suitably large $\sigma_t(F) : S^{N-1} \times \mathbb{T}^N \to GL_k(\mathbb{C})$ can be homotoped (through maps with image in $GL_{k'}(\mathbb{C})$) to a map

$$\sigma_t(F)' : S^{N-1} \times \mathbb{T}^N \to GL_N(\mathbb{C}) \oplus \{1_{k'-N}\} \subseteq GL_{k'}(\mathbb{C}),$$

where $k' = \max\{k, N\}$, and $1_{k'-N}$ is the identity in $k' - N$ dimensions (see [1, page 239]). Having performed this homotopy, $Index(F)$ can be computed using Theorem 6.1 above above, whatever the original k is. Moreover, as the degree of the map $\widetilde{\sigma_t(F)}$ is easily seen to be zero whenever $k < N$ (essentially as the image ends up being in a lower dimensional sphere), one has the following corollary.

Corollary 6.2. *There exist Fredholm operators in $\mathcal{A}(SO_k(\mathbb{Z}^N))$ of non-zero index if and only if $k \geq N$.*

Proof. One half of this has already been proved. For the existence of Fredholm operators of index one for all $k \geq N$, it is sufficient to note that there are maps $S^{N-1} \times \mathbb{T}^N \to S^{2N-1}$ of degree $(N-1)!$ coming from maps $S^{N_1} \times \mathbb{T}^N \to GL_N(\mathbb{C})$ (cf. [1, page 239] – this statement follows easily from results stated there), and use Theorem 6.1. □

Remark 6.3. On the other hand, there are Fredholm BDOs with non-trivial index, *non*-slowly-oscillating coefficients, and with *scalar* values on $l^2(\mathbb{Z}^N)$ for all N. Indeed, let P be the orthogonal projection onto the closed subspace of $l^2(\mathbb{Z}^N)$ spanned by

$$\{\delta_{m=(m_1,\dots,m_N)} : m_1 \leq 0, m_2 = m_3 = \cdots = m_N = 0\},$$

and let U be the unitary shift defined on basis elements by

$$U : \delta_{m_1,\dots,m_N} \mapsto \delta_{m_1+1,m_2\dots,m_N}.$$

Then $PUP + (1 - P)$ is a Fredholm band operator of index one.

The above corollary is thus of interest as it suggests that any index theorem applying to *all* BDOs on $\mathbb{Z}^N$ would have to be significantly different from Theorems 5.4 and 6.1. Note that this 'dimension problem' does not apply in the case $N = 1$ studied in [14] and [16].

Recall finally that Rabinovich and Roch [13] have used the results of [14] to prove an index theorem for locally compact operators on $\mathbb{R}^N$. We can partially extend their theorem using our machinery; a brief sketch is as follows.

There is a natural definition of locally compact BDO with slowly oscillating coefficients on $\mathbb{R}^N$; one can moreover show that the C^*-algebra of such operators is isomorphic to $\bar{\mathfrak{c}}(\mathbb{Z}^N) \rtimes_r \mathbb{Z}^N$. Such operators are *rich* in the sense of [15] if and only if they are actually in the (much smaller) subalgebra $C(\overline{\mathbb{Z}^N}^h, \mathcal{K}) \rtimes_r \mathbb{Z}^N$. Fredholm operators in this algebra can be handled in exactly the same way as those in $\mathcal{A}(SO_k(\mathbb{Z}^N)) \cong C(\overline{\mathbb{Z}^N}^h, M_k(\mathbb{C})) \rtimes_r \mathbb{Z}^N$, thus proving an index theorem precisely analogous to Theorems 5.4 and 6.1 above.

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