Math 3430
Existence and Uniqueness Worksheet
February 8, 2021

Differential equations *rarely* admit explicit solutions in terms of known functions. Numerical approximations to solutions and numerical integration are alternative ways to get answers to physical phenomena that are modeled by differential equations. But what if a solution to an IVP *isn't* unique? There could be multiple solutions, or worse, maybe there aren't any solutions at all, even theoretically! Then even numerical approaches become moot. :(Let's investigate when these things happen.

1. Consider the following IVP:

$$\frac{dy}{dt} = -2\sqrt{y}, \quad y(1) = 0. \tag{1}$$

   (a) Without using (almost) any calculus, write a simple function $y(t)$ that satisfies the IVP.

   (b) Use separation of variables to find the *general* solution (i.e. don't take into account the initial condition yet).

   (c) Now input your general solution above into the ODE. Be careful! For any real number x, we have that $\sqrt{x^2} = |x|$. This will impose a condition between the independent variable t and the constant of integration c. Rewrite your solution as a piecewise function that is defined for all $-\infty < t < \infty$.

   (d) What value(s) of c allow your general solution to satisfy the initial condition?

   (e) How many different solutions are there to this IVP?

The previous problem explores how uniqueness can fail. Now let's explore how to determine *existence* of solutions. To do this, we need to recast the existence question as a limit question. Just how the sequence $\{1/n\}_{n=1}^{\infty}$ has a limit of zero as $n \to \infty$, we

want to find a sequence of functions $\{y_n(t)\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} y_n(t) = y(t)$ where $y(t)$ is the solution to a general IVP for some $t \in (\alpha, \beta)$ (see below).

2. Consider the general IVP:

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0. \tag{2}$$

We can rewrite this as an *integral equation* by simply integrating both sides and invoking the fundamental theorem of calculus:

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds. \tag{3}$$

From here we define the **Picard Iterates** as the elements $y_n(t)$ of a sequence of functions given by

$$y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_n(s))\, ds, \tag{4}$$

where $y_0(t) = y_0$ is given by the initial condition.

(a) Verify that equation (2) and equation (3) are equivalent.

(b) Find $y_4(t)$ if the IVP is

$$\dot{y} = y, \quad y(0) = 1. \tag{5}$$

(c) Can you guess a pattern for $y_n(t)$ for the IVP stated in part (b)?

(d) What is $\lim_{n\to\infty} y_n(t)$? Verify that it satisfies the IVP from part (b).

Now we state the important theorem about existence and uniqueness of solutions for IVPs. This theorem applies only for 1st order scalar ODEs, but can be easily adapted to 1st order systems, and by extension, higher order ODE systems as well.

Theorem 1. *Consider the IVP:*

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0. \tag{6}$$

Existence: *If $f(t, y)$ is continuous as a function of two variables on an open set $S \subseteq \mathbb{R}^2$ that contains the point (t_0, y_0), then there is a solution, $y(t)$, of the ODE such that $y(t_0) = y_0$ that is defined on some interval (α, β) containing t_0.*

Uniqueness: *If $\dfrac{\partial f}{\partial y}$ exists and is continuous on S, then the solution $y(t)$ is unique.*

Remark 1. *There are more general versions of this theorem, in particular the case where the partial derivative requirement is replaced by Lipschitz continuity (Picar-Lindelöf Theorem). There is also the Carathéodory Theorem, which can handle cases when $f(t, y)$ is not continuous, if you are willing to pay the price of having solutions which are not continuous in a traditional sense (which probably sounds odd since we need differentiability!).*