Math 3430
Series Solutions Worksheet
October 21, 2020

1. Find the recurrence relations that determine the coefficients of the series solutions centered at $t = 0$ for the following ODEs:

   (a) $y'' - y' - y = 3t^2$ **Solution:**
   Take it term by term:

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n, \\ y'(t) &= \sum_{n=1}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} t^n, \\ y''(t) &= \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n. \end{aligned} \tag{1}$$

   Now combining these equations as per the ODE we find

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - (n+1) a_{n+1} - a_n \right] t^n = 3t^2. \tag{2}$$

   If the right-hand side of the ODE were zero, then we could deduce the following recurrence realtion

$$a_{n+2} = \frac{a_{n+1}}{n+2} + \frac{a_n}{(n+1)(n+2)} \tag{3}$$

   for all $n \geq 0$; however, because the right-hand side is nonzero, we need to compare our series from the left-hand side to the "series" from the right-hand side. In the case that $n = 0, 1$ and 2 we have

$$\begin{aligned} a_2 &= \frac{1}{2} a_1 + \frac{1}{2} a_0 \quad & a_3 &= \frac{1}{3} a_2 + \frac{1}{6} a_1 = \frac{1}{3} a_1 + \frac{1}{6} a_0 \\ & & a_4 &= \frac{1}{4} a_3 + \frac{1}{12} a_2 + 3 = \frac{1}{8} a_1 + \frac{1}{12} a_0 + 3 \end{aligned} \tag{4}$$

$$a_{n+2} = \frac{a_{n+1}}{n+2} + \frac{a_n}{(n+1)(n+2)}. \tag{5}$$

   when $n \geq 2$.

   (b) $(1 + t^4)y'' - ty' + 3t^2 y = 0$
   Solution:

We will distribute y'' over $(1+t^4)$. Now take it term by term again:

$$\begin{aligned} 3t^2y(t) &= \sum_{n=0}^{\infty} 3a_n t^{n+2}, \\ -ty'(t) &= -\sum_{n=1}^{\infty} na_n t^n = -\sum_{n=-1}^{\infty} (n+2)a_{n+2}t^{n+2}, \\ y''(t) &= \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} = \sum_{n=-2}^{\infty} (n+4)(n+3)a_{n+4}t^{n+2}, \\ t^4y''(t) &= \sum_{n=2}^{\infty} n(n-1)a_n t^{n+2}. \end{aligned} \tag{6}$$

Notice that once all the series are set-up to have a highest power of $n+2$ that we have all kinds of different initial counts on each one. We want to pick the starting position for the index among all the series and that means we must "pull out" the earlier terms. This gives

$$\begin{aligned} &2a_2 + (6a_3 - a_1)t + (12a_4 - 2a_2 + 3a_0)t^2 + (20a_5 - 3a_3 + 3a_1)t^3 + \\ &+ \sum_{n=2}^{\infty} \left[(n+4)(n+3)a_{n+4} - (n+2)a_{n+2} + (n^2 - n + 3)a_n\right] t^{n+2} = 0. \end{aligned} \tag{7}$$

Thus we have

$$\begin{aligned} a_2 &= 0, \qquad a_3 = \frac{1}{6}a_1, \\ a_4 &= \frac{1}{4}a_0, \quad a_5 = \frac{1}{8}a_1, \\ a_{n+4} &= \frac{(n+2)a_{n+2}}{(n+4)(n+3)} - \frac{(n^2-n+3)a_n}{(n+4)(n+3)}, \text{ for } n \geq 2. \end{aligned} \tag{8}$$

2. Find the general solution of the following ODEs:

   (a) $y'' + 2ty' + 2y = 0$

   Solution:

   Once again, we'll take it term by term:

$$\begin{aligned} 2y(t) &= \sum_{n=0}^{\infty} 2a_n t^n, \\ 2ty'(t) &= \sum_{n=1}^{\infty} 2na_n t^n, \\ y''(t) &= \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}t^n. \end{aligned} \tag{9}$$

   Substituting these into the ODE we find

$$2(a_0 + a_2) + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} + 2(n+1)a_n\right] t^n = 0, \tag{10}$$

and therefore the recurrence relation is

$$a_{n+2} = -\frac{2a_n}{n+2}, \quad n \geq 0 \tag{11}$$

Note: the constant term from eq. (10) agrees with the recurrence relation for the higher a_n terms. This is not always true, as we have seen in previous problems.

To solve this recurrence relation, we notice that the index is skipping by two, thus the even and odd terms will be determined separately. For the even terms, we get

$$a_{2k} = (-1)^k \frac{2^k}{(2k)!!} a_0, \qquad k \geq 0$$

and the odd terms will be

$$a_{2k+1} = (-1)^k \frac{2^k}{(2k+1)!!} a_1, \qquad k \geq 0.$$

Thus a general solution for this ODE is

$$y(t) = a_0 \sum_{n=0}^{\infty} (-1)^n \frac{2^n}{(2n)!!} t^{2n} + a_1 \sum_{n=0}^{\infty} (-1)^n \frac{2^n}{(2n+1)!!} t^{2n+1} \tag{12}$$

where a_0 and a_1 are arbitrary constants.

(b) $y'' - 2ty' + 2my = 0$ where m is a positive integer.
Solution:
Once again, we'll take it term by term:

$$\begin{aligned} 2my(t) &= \sum_{n=0}^{\infty} 2ma_n t^n, \\ -2ty'(t) &= -\sum_{n=1}^{\infty} 2na_n t^n, \\ y''(t) &= \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n. \end{aligned} \tag{13}$$

Substituting these into the ODE we find

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + 2(m-n)a_n \right] t^n = 0, \tag{14}$$

and therefore the recurrence relation is

$$a_{n+2} = \frac{2(n-m)a_n}{(n+1)(n+2)}. \tag{15}$$

Notice that again the recurrence relation skips by two on the index of the a_n terms; however, it seems something strange might happen when $n = m$. We need to look at two cases now: 1) $m = 2l$ for some $l \geq 1$ (i.e. m is even) or 2) $m = 2l + 1$ for some $l \geq 1$ (i.e. m is odd). Let's start with the even case of m.

Now we have to sub-divide into two cases again, but this time for even n or odd n. Starting with odd n we find that

$$\begin{aligned}
a_{2k+1} &= \frac{(-1)^k 2^k (2l-1)!!}{(2l-2k-1)!!(2k+1)!} a_1, \qquad 0 \leq k \leq l-1, \\
a_{2l+1} &= \frac{-2}{(2l)(2l+1)} a_{2l-1}, \\
&= \frac{(-1)^l 2^l (2l-1)!!}{(2l+1)!} a_1, \qquad k = l \\
a_{2k+1} &= \frac{2(2k-2l-1)}{(2k)(2k+1)} a_{2k-1} \\
&= \frac{-2^k (2k-2l-1)!!(2l-1)!}{(2k+1)!} a_{2l-1} \\
&= \frac{(-1)^l 2^{k+l-1} (2k-2l-1)!!(2l-1)!!}{(2k+1)!} a_1, \qquad k \geq l+1.
\end{aligned}$$

For n even we find

$$\begin{aligned}
a_{2k} &= \frac{(-1)^k 2^k (2l)!!}{(2l-2k)!!(2k)!} a_0, \qquad 0 \leq k \leq l-1, \\
a_{2k} &= 0, \qquad k \geq l.
\end{aligned}$$

So one of our "series" is actually a finite polynomial! Let's write the general solution for this case of m even.

$$\begin{aligned}
y(t) = a_1 \sum_{n=0}^{l-1} \frac{(-1)^n 2^n (2l-1)!!}{(2l-2n-1)!!(2n+1)!} t^{2n+1} + a_1 \frac{(-1)^l 2^l (2l-1)!!}{(2l+1)!} t^{2l+1} \\
+ a_1 \sum_{n=l+1}^{\infty} \frac{(-1)^l 2^{n+l-1} (2n-2l-1)!!(2l-1)!!}{(2n+1)!} t^{2n+1} + a_0 \sum_{n=0}^{l-1} \frac{(-1)^n 2^n (2l)!!}{(2l-2n)!!(2n)!} t^{2n}.
\end{aligned}$$

where a_0 and a_1 are arbitrary constants. For the case of $m = 2l + 1$ the solution is very similar. Indeed, we should get

$$\begin{aligned}
y(t) = a_0 \sum_{n=0}^{l-1} \frac{(-1)^n 2^n (2l)!!}{(2l-2n)!!(2n+1)!} t^{2n} + a_0 \frac{(-1)^l 2^l (2l)!!}{(2l+2)!} t^{2l} \\
+ a_0 \sum_{n=l}^{\infty} \frac{(-1)^l 2^{n+l-1} (2n-2l-2)!!(2l)!!}{(2n+1)!} t^{2n} + a_1 \sum_{n=0}^{l-1} \frac{(-1)^n 2^n (2l+1)!!}{(2l-2n+1)!!(2n)!} t^{2n+1}.
\end{aligned}$$

Warning: I have not checked the above solution for odd m very closely, but it should be attainable from the previous solution by replacing $2l$ with $2l+1$ and being cognizant about the odd and even powers of t.

3. In this exercise we'll find the general solution to Bessel's differential equation:

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - \alpha^2)y = 0, \tag{16}$$

where $\alpha > 0$.

(a) Attempt to find a series solution to Bessel's ODE with the usual approach, i.e. assume $y(t) = \sum_{n=0}^{\infty} a_n t^n$. What happens in this case? Does this violate our theorem about solving ODEs via power series? Why or why not?
Answer: You'll get the zero series! It doesn't violate our theorem since the functions $p(t) = \frac{1}{t}$ and $q(t) = \frac{t^2 - \alpha^2}{t^2}$ are not analytic around $t = 0$.

(b) To find a series solution, you will have to modify the series a bit. Let $y(t) = \sum_{n=0}^{\infty} a_n t^{n+\lambda}$ where λ is any nonzero real number. Find the recurrence relations by making this assumption about the form of the solution. Make sure to pay special attention to the t^λ and $t^{\lambda+1}$ terms.
Solution: Take it term by term as usual: **Warning:** Since the powers aren't necessarily integers, we needn't worry about having negative powers of t anymore. So we can keep the starting index the same in all cases.

$$\begin{aligned}
t^2y(t) &= \sum_{n=0}^{\infty} a_n t^{n+\lambda+2}, \\
-\alpha^2 y(t) &= -\sum_{n=0}^{\infty} \alpha^2 a_n t^{n+\lambda} = -\sum_{n=-2}^{\infty} \alpha^2 a_{n+2} t^{n+\lambda+2}. \\
ty'(t) &= \sum_{n=0}^{\infty} (n+\lambda) a_n t^{n+\lambda} = \sum_{n=-2}^{\infty} (n+\lambda+2) a_{n+2} t^{n+\lambda+2}, \\
t^2y''(t) &= \sum_{n=0}^{\infty} (n+\lambda)(n+\lambda-1) a_n t^{n+\lambda} = \sum_{n=-2}^{\infty} (n+\lambda+2)(n+\lambda+1) a_{n+2} t^{n+\lambda+2},
\end{aligned} \tag{17}$$

Adding up all of the above, and then pulling out the first one or two terms for each series that doesn't start at $n = 0$ we find

$$(\lambda^2-\alpha^2)a_0 t^\lambda + ((\lambda+1)^2-\alpha^2)a_1 t^{\lambda+1} + \sum_{n=0}^{\infty} \left[((n+\lambda+2)^2 - \alpha^2)a_{n+2} + a_n \right] t^{\lambda+n} \tag{18}$$

So we have the relations

$$\begin{aligned}
(\lambda^2 - \alpha^2)a_0 &= 0 \\
((\lambda+1)^2 - \alpha^2)a_1 &= 0 \\
a_{n+2} &= -\frac{a_n}{(n+\lambda+2)^2 - \alpha^2}
\end{aligned} \tag{19}$$

(c) Solve the recurrence relation you found in the previous question.