The Zeros of Derivatives of Entire Functions and the Pólya-Wiman Conjecture

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The zeros of derivatives of entire functions and the Pólya-Wiman Conjecture

By Thomas Craven, George Csordas and Wayne Smith

1. Introduction

The purpose of this paper is to prove the following fifty-five year-old conjecture of Pólya [P2], [P4] and Wiman [Wi1], [Wi2].

**The Pólya-Wiman Conjecture.** If a real entire function $f(x)$ of order less than two has only a finite number of nonreal zeros, then there is a positive integer $m_0$ such that if $m \geq m_0$, then $f^{(m)}(x)$ has only real zeros.

An entire function $f(x)$ which assumes only real values on the real axis is said to be a real entire function. Thus, if $f$ is a real entire function, then its Maclaurin coefficients are all real, and consequently the zeros of $f$ are symmetrically located with respect to the real axis. One of the intuitive principles underlying the theory of distribution of zeros of successive derivatives of real entire functions was formulated by Pólya [P4] in his celebrated survey article entitled *On the zeros of the derivatives of a function and its analytic character*. In this work [P4, p. 181] Pólya states, "The real axis seems to exert an influence on the complex zeros of $f^{(n)}(z)$; it seems to attract these zeros when the order is less than 2, and it seems to repel them when the order is greater than 2." Indeed, the first confirmation of this principle was made by Ålander [A2] (see also Pólya [P2]) who in 1930 showed that the Pólya-Wiman conjecture is true if the order of $f(x)$ is less than 2/3. In [Wi1] and [Wi2] Wiman established the validity of this conjecture for functions $f(x)$ of order at most 1, while in 1937 Pólya [P3] proved it for functions of order less than 4/3. A general survey of these results and related conjectures may be found in [P2] and [P4]. (For more recent surveys of this and related areas of research see Boas [B2] and Prather [Pr].) It seems that no progress has been made on the Pólya-Wiman conjecture since the publication of Pólya's 4/3 theorem in 1937.

Our proof of the conjecture is self-contained and does not rely on the results cited above. In Section 2 we will (1) introduce some definitions and notations, (2) state some of the properties of functions in the Laguerre-Pólya class, and (3) prove the Pólya-Wiman conjecture (Theorem 3). In the proof of Theorem 3 we
will require two technical results, Theorem 1 and Theorem 2, whose verification will be postponed until Section 3. The paper concludes with some remarks on Theorem 3 and some open problems (Section 4).

2. The proof of the Pólya-Wiman conjecture

We begin this section with some nomenclature and with a brief review of those properties of functions in the Laguerre-Pólya class which will be used in the sequel.

A real entire function \( \psi(x) \) is said to be in the Laguerre-Pólya class if \( \psi(x) \) can be expressed in the form

\[
\psi(x) = c x^n e^{-ax^2 + bx} \prod_{k=1}^{\infty} (1 + x/a_k) e^{-x/a_k},
\]

where \( c, \beta, a_k \) are real, \( \alpha \geq 0 \), \( n \) is a nonnegative integer and \( \sum a_k^{-2} < \infty \). If \( \psi(x) \) is in the Laguerre-Pólya class, we will write \( \psi \in L - P \). By the classical results of Laguerre [L] and Pólya [P1] (see also Lindwart and Pólya [LP]), \( \psi \in L - P \) if and only if \( \psi \) can be uniformly approximated on disks about the origin by a sequence of polynomials with only real zeros. (For a modern proof of this theorem see Levin [Le, Chapter 8].) Thus, it follows from this result that the class \( L - P \) is closed under differentiation; that is, if \( \psi \in L - P \), then \( \psi^{(n)} \in L - P \) for \( n \geq 0 \). Moreover, any easy calculation shows that the logarithmic derivative of a function \( \psi(x) \) in \( L - P \), \( \psi(x) \neq c e^{ax} \), is strictly decreasing:

\[
D \left( \frac{\psi'}{\psi} (x) \right) < 0, \quad x \in \mathbb{R}, \quad D = \frac{d}{dx}.
\]

Here and in the sequel it will be frequently convenient to denote differentiation by \( D \).

A function \( \varphi(x) \) is said to be in \( L - P^* \), written \( \varphi \in L - P^* \), if \( \varphi(x) \) can be expressed in the form

\[
\varphi(x) = p(x) \psi(x),
\]

where \( p(x) \in \mathbb{R}[x] \); that is, \( p(x) \) is a real polynomial, and \( \psi(x) \in L - P \). Thus if \( \varphi \in L - P^* \), then \( \varphi \) has at most a finite number of nonreal zeros. If \( \varphi \in L - P^* \), then \( Z_c(\varphi) \) will denote the number of nonreal zeros of \( \varphi \), counting multiplicities. From the above facts and Rolle's theorem it follows that

\[
Z_c(\varphi'(x)) \leq Z_c(\varphi(x)).
\]

If \( f \) is a meromorphic function having only a finite number of real zeros, then \( Z_R(f) \) will denote the number of real zeros of \( f \), counting multiplicities.

Finally, the order \( \rho \) of an entire function \( \varphi \) will be denoted by \( \rho(\varphi) \).
Our proof of the Pólya-Wiman conjecture will be based on two lemmas and the following two technical results, Theorem 1 and Theorem 2, whose proofs will be presented in Section 3.

**Theorem 1.** Let \( \varphi \in \mathcal{L} - \mathcal{P}^* \). Suppose that \( \rho(\varphi) < 2 \) and that \( \varphi \) has exactly \( 2d \), \( d > 0 \), nonreal zeros. Let \( \gamma \in \mathbb{R} \). Then the following statements are equivalent.

(a) \((D + \gamma)\varphi(x) \in \mathcal{L} - \mathcal{P}, \ D = d/dx. \)
(b) (i) \( Z_{\mathbb{R}}(D(\varphi'/\varphi)) = 2d, \) and
   (ii) if \( x_1 \leq x_2 \leq \cdots \leq x_{2d} \) are the real zeros of \( D(\varphi'/\varphi) \) then \( \varphi'/\varphi(x_{2j-1}) \leq -\gamma \) and \( \varphi'/\varphi(x_{2j}) \geq -\gamma, \) \( 1 \leq j \leq d, \) and \( \varphi(x) \neq 0 \) for \( x_{2j-1} \leq x \leq x_{2j}, \) \( 1 \leq j \leq d. \)

As an immediate consequence of Theorem 1, we obtain the following corollary.

**Corollary 1.** Let \( \varphi \in \mathcal{L} - \mathcal{P}^* \) and suppose that \( \rho(\varphi) < 2 \). If \( \gamma_1 < \gamma_2 \) and if \( (D + \gamma_j)\varphi(x) \in \mathcal{L} - \mathcal{P}, \ j = 1, 2, \) then

\[(D + \gamma)\varphi(x) \in \mathcal{L} - \mathcal{P} \quad \text{for all } \gamma \in [\gamma_1, \gamma_2].\]

Moreover, the real zeros of \( D(\varphi'/\varphi) \) are all simple.

**Theorem 2.** Let \( \varphi \in \mathcal{L} - \mathcal{P}^* \) and suppose that \( \rho(\varphi) < 2 \). If \( (D + \gamma)\varphi(x) \in \mathcal{L} - \mathcal{P} \) for all \( \gamma \) in an open nonempty interval \( I \), then there is a positive integer \( m \) such that \( D^m\varphi(x) \in \mathcal{L} - \mathcal{P} \).

**Lemma 1.** If \( \varphi \in \mathcal{L} - \mathcal{P}^* \) and if \( D^m\varphi \in \mathcal{L} - \mathcal{P} \) for some nonnegative integer \( m \), then for any \( a \in \mathbb{R} \)

\[(2.4) \quad D^{m+1}[(x + a)\varphi(x)] \in \mathcal{L} - \mathcal{P}. \]

**Proof.** The proof is an immediate consequence of the fact that the class \( \mathcal{L} - \mathcal{P} \) is closed under differentiation and the following observation.

\[
D(x + a)^{m+1}D^m\varphi(x) = (x + a)^m[(m + 1)D^m\varphi(x) + (x + a)D^{m+1}\varphi(x)]
\]

\[
= (x + a)^mD^{m+1}[(x + a)\varphi(x)]. \quad \Box
\]

**Lemma 2.** Let \( \varphi(x) = p(x)\psi(x) \in \mathcal{L} - \mathcal{P}^* \), where \( p(x) \in \mathbb{R}[x] \), \( \deg p = 2d > 0 \) and \( Z_c(p) = 2d \) and where

\[(2.5) \quad \psi(x) = cx^ne^{\beta x} \prod_{k=1}^{\infty} (1 + x/a_k)e^{-x/a_k}, \]

\( c, \beta, a_k \in \mathbb{R}, k = 1, 2, 3, \ldots \) and \( \Sigma a_k^{-2} < \infty \). Then there are a positive integer \( N \)
and an open nonempty interval $I$ such that if $\gamma \in I$, then
\begin{equation}
Z_c((D + \gamma)\varphi_N(x)) = 2s < 2d,
\end{equation}
where
\begin{equation}
\varphi_N(x) = cp(x) \left( \exp \left( \beta - \sum_{k=1}^{N-1} \frac{1}{a_k} \right) x \right) \prod_{k=N}^{\infty} (1 + x/a_k)e^{-x/a_k}.
\end{equation}

Proof. Since $\deg p = 2d > 0$, there exist three numbers $\varepsilon > 0$, $b_1$ and $b_2$ with $b_1 < 0 < b_2$ such that
\begin{equation}
\frac{p'(b_1)}{p}(b_1) < -\varepsilon \quad \text{and} \quad \frac{p'(b_2)}{p}(b_2) > \varepsilon.
\end{equation}
Next we choose a positive integer $N$ such that
\begin{equation}
|a_k| > \max \{ b_2, -b_1 \}, \quad \text{for all } k \geq N,
\end{equation}
and
\begin{equation}
\max \left\{ \sum_{k=N}^{\infty} \frac{b_2}{a_k(a_k + b_2)}, \sum_{k=N}^{\infty} \frac{-b_1}{a_k(a_k + b_1)} \right\} < \frac{\varepsilon}{2}.
\end{equation}
If $\psi(x)$ has only a finite number of zeros, say $a_1, \ldots, a_m$, then we drop the requirements (2.9) and (2.10) and just set $N = m + 1$. In this case the canonical product appearing in (2.7) reduces to 1, and we replace (2.10) with the inequality $0 < \varepsilon/2$. Now let
\begin{equation}
I = \left\{ t \in \mathbb{R} : \left| \beta - \sum_{k=1}^{N-1} \frac{1}{a_k} \right| + t \right| < \frac{\varepsilon}{2} \right\},
\end{equation}
and let $\gamma \in I$. Having selected the positive integer $N$, we form the function $\varphi_N(x)$ given by (2.7). Next, we approximate $e^{\gamma x}\varphi_N(x)$ by the function
\begin{equation}
q_r(x) = p(x) \left( \exp \left( \beta - \sum_{k=1}^{N-1} \frac{1}{a_k} + \gamma \right) x \right) \prod_{k=N}^{N+r} (1 + x/a_k)e^{-x/a_k}.
\end{equation}
Then
\begin{equation}
\frac{q'_r}{q_r}(x) = \frac{p'}{p}(x) + \left( \beta - \sum_{k=1}^{N-1} \frac{1}{a_k} + \gamma \right) + \sum_{k=N}^{N+r} \frac{-x}{a_k(a_k + x)}
\end{equation}
and hence by (2.8), (2.9), (2.10) and (2.11),
\begin{equation}
\frac{q'_r}{q_r}(b_1) < 0 \quad \text{and} \quad \frac{q'_r}{q_r}(b_2) > 0.
\end{equation}
Consequently, it follows that the function $q'_r/q_r(x)$ has at least 2 more real zeros than the number of zeros guaranteed by Rolle's theorem and multiplicity considerations. Since $q_r(x)$ has $r + 1$ real zeros, $q'_r(x)$ has at least $r + 2$ real zeros. Now set

$$t_N = \beta - \sum_{k=1}^{N+r} \frac{1}{a_k} + \gamma.$$ 

Then $q'_r(x)$ has $2d + r + 1$ zeros if $t_N \neq 0$ and $2d + r$ zeros if $t_N = 0$. But then it follows that $q'_r(x)$ has at most $2d + r + 1 - (r + 2)$ or $2d + r - (r + 2)$ nonreal zeros. Therefore, $Z_c(q'_r(x)) = 2s \leq 2d - 2$. Finally, we let $r \to \infty$ and we invoke Hurwitz's theorem to conclude that for $\gamma \in I,$

$$Z_c(D(e^{\gamma x}\varphi_N(x))) = Z_c((D + \gamma)\varphi_N(x)) \leq 2d - 2. \quad \Box$$

**Remark 2.1.** If $\varphi \in \mathcal{L} - \mathcal{P}^*$ and if $\rho(\varphi) < 1$, then it follows from Lemma 1 and the proof of Lemma 2 (with $\gamma = 0$) that there is a positive integer $m$ such that $D^m\varphi(x) \in \mathcal{L} - \mathcal{P}$.

**Theorem 3.** Let $\varphi(x) \in \mathcal{L} - \mathcal{P}^*$ and suppose that $\rho(\varphi) < 2$. Then there is a positive integer $M$ such that $D^M\varphi(x) \in \mathcal{L} - \mathcal{P}$.

**Proof.** Let $\varphi(x) = p(x)\psi(x) \in \mathcal{L} - \mathcal{P}^*$, where $p(x)$ is a real polynomial of degree $2d$, $d > 0$, having exactly $2d$ nonreal zeros and where $\psi(x)$ is defined by (2.5).

The following proof is based on induction on $d$. If $d = 1$, then by Lemma 2 there is a positive integer $N$ and open interval $I$ such that if $\gamma \in I$, then $(D + \gamma)\varphi_N(x) \in \mathcal{L} - \mathcal{P}$, where $\varphi_N(x)$ is given by (2.7). But then by Theorem 2 there is a positive integer $m$ such that $D^m\varphi_N(x) \in \mathcal{L} - \mathcal{P}$. Since $\varphi(x) = x^n(\prod_{k=1}^{N-1}(1 + x/a_k))\varphi_N(x)$, we can apply Lemma 1 to conclude that $D^m + m - 1 + n\varphi(x) \in \mathcal{L} - \mathcal{P}$.

We next assume that the theorem is true for $1 \leq d \leq k - 1$ and that $Z_c(\varphi) = 2k$. Then by Lemma 2 there are a positive integer $N$ and an open nonempty interval $I$ such that if $\gamma \in I$, then

$$Z_c((D + \gamma)\varphi_N(x)) \leq 2(k - 1), \quad (2.13)$$

where $\varphi_N(x)$ is given by (2.7). Now let $\gamma_1, \gamma_2 \in I$, $\gamma_1 < \gamma_2$. Then since (2.13) holds with $\gamma$ replaced by $\gamma_1$ or $\gamma_2$, we can invoke the induction hypothesis to assert that there are positive integers $m_1$ and $m_2$ such that $D^{m_1}(D + \gamma_1)\varphi_N(x) \in \mathcal{L} - \mathcal{P}$ and $D^{m_2}(D + \gamma_2)\varphi_N(x) \in \mathcal{L} - \mathcal{P}$. Hence if we set $m = \max(m_1, m_2)$, then

$$D^m(D + \gamma_j)\varphi_N(x) \in \mathcal{L} - \mathcal{P}, \quad j = 1, 2.$$
But then by Corollary 1
\[ D^m(D + \gamma)\varphi_N(x) = (D + \gamma)D^n\varphi_N(x) \in \mathcal{L} - \mathcal{P} \]
for all \( \gamma \in \mathcal{I} = (\gamma_1, \gamma_2) \). Hence by Theorem 2 there is a positive integer \( r \) such that \( D^{r+m}\varphi_N(x) \in \mathcal{L} - \mathcal{P} \). Finally, we apply Lemma 1 to conclude that \( D^{r+m+N-1+n}\varphi(x) \in \mathcal{L} - \mathcal{P} \). \( \square \)

3. The proofs of Theorem 1 and Theorem 2

Broadly speaking, the proofs of Theorem 1 and Theorem 2 are based on two general principles. One of these involves the location of the nonreal zeros of \( \varphi^{(k)}(x) \), where \( \varphi \in \mathcal{L} - \mathcal{P}^* \). The other principle expresses a relationship involving the number of real zeros of \( \varphi^{(k)}(x) \) on certain intervals and the number of nonreal zeros of \( \varphi^{(k)}(x) \).

For the reader's convenience, we recall in this section some of those properties of the zeros of functions in \( \mathcal{L} - \mathcal{P}^* \) which will be used in the sequel.

Let \( \alpha \pm i\beta \), with \( \alpha, \beta \in \mathbb{R} \), denote a pair of conjugate nonreal zeros of \( \varphi \in \mathcal{L} - \mathcal{P}^* \). Consider the ellipse whose minor axis has \( \alpha + i\beta \) and \( \alpha - i\beta \) as endpoints and whose major axis has the length \( 2\beta\sqrt{k} \), \( k \geq 1 \). This ellipse is called a Jensen ellipse of order \( k \) of \( \varphi(x) \). We will refer to a Jensen ellipse of order one together with its interior as a Jensen disk. We will now state a theorem which, for real polynomials, was announced without proof by Jensen [J] in 1913. It was proved by Walsh [W] in 1920 and later by Nagy [N1].

**Theorem (The Jensen-Nagy-Walsh theorem).** Let \( \varphi \in \mathcal{L} - \mathcal{P}^* \). Then every nonreal zero of \( \varphi^{(k)}(x) \) lies inside or on at least one of the Jensen ellipses of order \( k \) of \( \varphi(x) \).

Let \( \varphi(x) \) be a nonconstant real entire function of finite order \( \rho \) and let \( \varphi(x) \) have exactly \( 2d \) nonreal zeros, \( d \geq 0 \). By Rolle's theorem \( \varphi'(x) \) has an odd number of real zeros (and a fortiori at least one) between any two consecutive real zeros, say \( a \) and \( b \), \( a < b \), of \( \varphi(x) \). Counting all zeros with their multiplicities, suppose that \( \varphi'(x) \) has \( 2k + 1 \) zeros between \( a \) and \( b \). Then we will say that \( \varphi'(x) \) has \( 2k \) extra zeros between \( a \) and \( b \). If \( \varphi(x) \) has a largest zero \( a_L \) (or a smallest zero \( a_s \)), then any real zero \( \varphi'(x) \) in \( (a_L, \infty) \) (or in \( (-\infty, a_s) \)) will be called an extra zero of \( \varphi'(x) \). The total number of extra zeros \( E(\varphi') \) counting multiplicities will be denoted by \( E(\varphi') \). Now by the classical theorem of Borel-Laguerre (see, for example, [M, p. 837]),
\[ E(\varphi') + Z_c(\varphi') \leq 2d + \lfloor \rho \rfloor, \]
where \( \lfloor \rho \rfloor \) denotes the largest integer not exceeding \( \rho \).
Remarks 3.1. (a) We first note that the multiple real zeros of $\varphi$ are \emph{not} counted as extra real zeros of $\varphi'$.

(b) Let $f(x)$ be a real polynomial, $\deg f = n \geq 1$, and suppose that $f$ has exactly $2d$, $d \geq 0$, nonreal zeros. Then an easy argument shows that

$$E(f') + Z_c(f') = \begin{cases} 2d & \text{if } n > 2d \\ 2d - 1 & \text{if } n = 2d. \end{cases}$$

(c) In the sequel it will be convenient for us to use the following terminology. If $a$ and $b$, $a < b$, are two consecutive zeros of a function $\varphi$ in $\mathcal{L} - \mathcal{P}^*$, then we will say that one zero, counting multiplicity, of $\varphi'$ in $(a, b)$ is guaranteed by Rolle's theorem.

Preliminaries aside, we will now establish a lemma which will serve as a valuable tool in our subsequent counting arguments.

Lemma 3. Let $\varphi$ be a transcendental function in $\mathcal{L} - \mathcal{P}^*$. If $\varphi$ has order $\rho$ and if $\varphi$ has exactly $2d$, $d \geq 0$, nonreal zeros then

$$2d \leq E(\varphi') + Z_c(\varphi') \leq 2d + [\rho],$$

where $[\rho]$ denotes the largest integer not exceeding $\rho$.

Proof. By the Borel-Laguerre theorem (cf. (3.1)) it suffices to establish that $E(\varphi') + Z_c(\varphi') \geq 2d$, when $d > 0$. Let $D_1, D_2, \ldots, D_d$ denote the closed Jensen disks associated with the nonreal zeros of $\varphi$. Now we select two real numbers $a$ and $b$ such that for $1 \leq j \leq d$, $D_j \cap \mathbb{R} \subseteq [a, b]$, where $a$ and $b$ are either zeros of $\varphi$ or if $\varphi$ has a smallest zero $a_s$, then $a < a_s$, and if $\varphi$ has a largest zero $a_L$, then $a_L < b$. Let

$$K = [a, b] \cup \left( \bigcup_{j=1}^{d} D_j \right).$$

Suppose that $\varphi(x) = p(x)\psi(x)$, where $\psi \in \mathcal{L} - \mathcal{P}$ and $p$ is a real polynomial of degree $2d$ with no real zeros. Since $\psi \in \mathcal{L} - \mathcal{P}$, we can find a sequence of nonconstant polynomials $\{q_n\}_{n=1}^{\infty}$, such that (1) $q_n$ has only real zeros, (2) in $K$, $q_n$ has the same real zeros, counting multiplicities, as $\psi$ has, and (3) $q_n \to \psi$ uniformly on compact subsets of $\mathbb{C}$ as $n \to \infty$. For $n \geq 1$, let $f_n(x) = p(x)q_n(x)$. Then $f'_n/f_n$ is strictly decreasing on $\mathbb{R} \setminus K$. This is a special case of a result established in Lemma 6 (see (3.11) and the subsequent remark). Thus, it follows that $f'_n$ has no extra real zeros outside $K$. Moreover, by the Jensen-Nagy-Walsh theorem, all the nonreal zeros of $f'_n$ lie in $K$. Thus, if $m$ real zeros of $f'_n$ in $K$ are guaranteed by Rolle's theorem (cf. Remark 3.1(c)), then by Remark 3.1(b), $f'_n$ has $m + 2d$ zeros in $K$. Since the sequence $\{f'_n\}$ converges uniformly on compact subsets of $\mathbb{C}$ to $\varphi'$, it follows from Rouché's theorem that
\( \varphi' \) has at least \( m + 2d \) zeros in \( K \). Since only \( m \) of these zeros are guaranteed by Rolle’s theorem, \( E(\varphi') + Z_c(\varphi') \geq 2d \), and the proof of the lemma is complete. \( \square \)

**Remark 3.2.** Since on a compact set \( K \) the number of real zeros of \( \varphi' \) which are not zeros of \( \varphi \) is equal to the number of real zeros of \( \varphi'/\varphi \), it makes sense to define

\[
E\left( \frac{\varphi'}{\varphi} \right) = E(\varphi').
\]

Similarly, we define, for \( \gamma \in \mathbb{R} \),

\[
E((D + \gamma)\varphi) = E(D(e^{\gamma x}\varphi)).
\]

**Theorem 1.** Let \( \varphi \in \mathcal{L} - \mathcal{P}^* \). Suppose that \( \rho(\varphi) < 2 \) and that \( \varphi \) has exactly \( 2d \), \( d > 0 \), nonreal zeros. Let \( \gamma \in \mathbb{R} \). Then the following statements are equivalent.

(a) \( (D + \gamma)\varphi(x) \in \mathcal{L} - \mathcal{P} \), \( D = d/dx \).

(b) (i) \( Z_R(D(\varphi'/\varphi)) = 2d \), and

(ii) if \( x_1 \leq x_2 \leq \cdots \leq x_{2d} \) are the real zeros of \( D(\varphi'/\varphi) \) then \( (\varphi'/\varphi)(x_{2j-1}) \leq -\gamma \) and \( (\varphi'/\varphi)(x_{2j}) \geq -\gamma \), \( 1 \leq j \leq d \), and \( \varphi(x) \neq 0 \) for \( x_{2j-1} \leq x \leq x_{2j} \), \( 1 \leq j \leq d \).

**Proof:** We first observe that we may assume without loss of generality that \( \gamma = 0 \). Indeed, if we set \( F(x) = e^{\gamma x}\varphi(x) \), then it is clear that \( F(x) \in \mathcal{L} - \mathcal{P}^* \), \( \rho(F) < 2 \) and \( Z_c(F) = 2d \). Moreover,

\[
D\left( \frac{F'}{F} \right) = D\left( \frac{\varphi'}{\varphi} + \gamma \right) = D\left( \frac{\varphi'}{\varphi} \right).
\]

Therefore, it suffices to show that (a) and (b) are equivalent when \( \gamma = 0 \).

I. We will first establish the theorem under the additional assumption that \( \varphi(x) \) has an infinite number of real zeros. In this case we may assume, without loss of generality, that \( \varphi \) has an infinite number of negative real zeros. (b) \( \implies \) (a): By Lemma 3

\[
E(\varphi') + Z_c(\varphi') = 2d \quad \text{or} \quad 2d + 1,
\]

and thus it suffices to prove that \( E(\varphi') \geq 2d \), or equivalently that \( E(\varphi'/\varphi) \geq 2d \) (cf. Remark 3.2 following Lemma 3). Let \( J = (a, b) \), where \( a \) and \( b \) are any two consecutive zeros of \( \varphi \) and let \( J_\infty = (c, \infty) \), if \( \varphi \) has a largest zero \( c \). Now it follows from assumption (b)(ii) that \( D(\varphi'/\varphi) \) has an even number (possibly 0) of zeros in \( J \) (in \( J_\infty \)). In addition, (b)(ii) also implies that if \( D(\varphi'/\varphi) \) has \( 2n \) zeros in
J (in \( J_{\infty} \)), then \( \varphi'/\varphi \) has at least \( 2n + 1 \) zeros in \( J \) (at least \( 2n \) zeros in \( J_{\infty} \)). Therefore, we conclude that \( \varphi'/\varphi \) has at least \( 2n \) extra zeros in \( J \) (in \( J_{\infty} \)). Thus, by (b)(i), \( E(\varphi'/\varphi) \geq 2d \).

(a) \( \Rightarrow \) (b)(i): The idea of the proof of this implication is to establish a precise relationship between the number of real zeros of \( D(\varphi'/\varphi) \) and the number of extra zeros of \( \varphi' \). The proof given below is divided into several parts. In Part (A) we provide the key device we will employ for counting zeros. In parts (B) and (C) we count the number of real zeros of \( D(\varphi'/\varphi) \) on certain bounded and unbounded intervals respectively. In part (D) we invoke Lemma 3 to conclude that \( Z_R(D(\varphi'/\varphi)) = 2d \).

(A) Let \( S \) denote the collection of all open intervals, \( I \), determined by pairs of consecutive zeros of \( \varphi'(x) \). Furthermore, assume that \( S \) also includes the interval \( I_{\infty} = (b_L, \infty) \) if \( \varphi' \) has a largest zero \( b_L \). If \( I \) is an interval in \( S \), then by Rolle's theorem \( \varphi \) has at most one zero in \( I \). By assumption \( \varphi' \in \mathcal{L} - \mathcal{P} \) and thus by Laguerre's theorem [B1, p. 23], for any real number \( \lambda \), the function

\[
\psi_\lambda(x) = (D + \lambda)\varphi'(x) = \varphi''(x) + \lambda\varphi'(x) = e^{-\lambda x}D(e^{\lambda x}\varphi'(x))
\]

has exactly one zero, counting multiplicity, in \( I \), if \( I \) is a bounded interval in \( S \); and \( \psi_\lambda(x) \) has at most one zero in \( I \), if \( I \) is an unbounded interval in \( S \). It now follows from Rolle's theorem that \( \lambda\varphi + \varphi' \) has no more than 2 zeros, counting multiplicity, on each of the intervals \( I \) in \( S \).

(B) Let \( I = (c, d) \), where \( c \) and \( d \) are two consecutive zeros of \( \varphi' \), and consider the following two cases. Case 1: \( \varphi \) has exactly one zero, call it \( a \), in \( I \), and case 2: \( \varphi \neq 0 \) in \( I \). In case 1 we may assume, without loss of generality, that \( \varphi > 0 \) on \( (c, a) \) and that \( \varphi < 0 \) on \( (a, d) \). Thus it follows that \( \varphi' < 0 \) on \( (c, d) \).

Now for an arbitrary nonzero real number \( \lambda \) define

\[
Q_\lambda(x) = \frac{\varphi'}{\varphi}(x) + \lambda.
\]

If \( \lambda > 0 \), then \( Q_\lambda(c) = \lambda > 0 \) and \( Q_\lambda(a) = -\infty \). In this case \( Q_\lambda(x) > 0 \) on \( (a, d) \) and so \( Q_\lambda(x) \) must have an odd number of zeros in \( (c, d) \). If \( \lambda < 0 \), then \( Q_\lambda(x) < 0 \) on \( (c, a) \) and \( Q_\lambda(x) \) has an odd number of zeros in \( (c, d) \). But by part (A), \( Q_\lambda(x) \) can have at most two zeros in \( I \), counting multiplicities. Consequently, \( Q_\lambda(x) \) has exactly one zero in \( (c, d) \). Now suppose, for the sake of argument, that \( Q_\lambda'(x) = D(\varphi'/\varphi)(x) = 0 \) at some point \( x_0 \) in \( I = (c, d) \). But then the function \( Q_\lambda(x) \), where \( \lambda = -\varphi'/\varphi(x_0) \), would have at least two zeros in \( I \), counting multiplicities. This contradiction shows that \( D(\varphi'/\varphi) \neq 0 \) in \( I = (c, d) \).
In case 2, \( \varphi(x) \neq 0 \) in \( I = (c, d) \). Thus, we may assume without loss of generality that \( \varphi' / \varphi(x) > 0 \) on \( I \). Since \( \varphi' / \varphi(c) = \varphi' / \varphi(d) = 0 \), it follows from Rolle’s theorem that \( \mathcal{D}(\varphi' / \varphi) \) has at least one zero in \( I \). Next, we suppose, for the sake of argument, that \( \mathcal{D}(\varphi' / \varphi) \) has two zeros, say \( x_1, x_2, x_1 \leq x_2 \), in \( I \). If \( x_1 = x_2 \), then \( x = x_1 \) is a multiple of zero of \( \mathcal{D}(\varphi' / \varphi) \), with multiplicity at least two. But then the function

\[
Q_\lambda(x) = \frac{\varphi'}{\varphi}(x) + \lambda, \quad \lambda = -\frac{\varphi'}{\varphi}(x_1),
\]

has a zero of multiplicity at least three at \( x = x_1 \). This contradicts the conclusion obtained in part (A) and hence \( x = x_1 \) must be a simple zero of \( \mathcal{D}(\varphi' / \varphi) \).

Next, assume that \( x_1 < x_2 \) and set \( \lambda_1 = \varphi' / \varphi(x_1) \) and \( \lambda_2 = \varphi' / \varphi(x_2) \). At the zero \( x_j \) of \( \mathcal{D}(\varphi' / \varphi) \),

\[
\frac{\varphi''}{\varphi'}(x_j) = \frac{\varphi'}{\varphi}(x_j), \quad j = 1, 2.
\]

Since \( \varphi' \in \mathcal{L} - \mathcal{P} \), \( (\varphi'' / \varphi') \) is strictly decreasing on \( I \) and hence \( \varphi'' / \varphi' \) is one-to-one on \( I \), so that \( \lambda_1 > \lambda_2 > 0 \).

Set

\[
Q_\lambda(x) = \frac{\varphi'}{\varphi}(x) + \lambda, \quad \lambda = -\lambda_2.
\]

Since \( Q_\lambda(x) \) is continuous on \( I \) and

\[-\lambda_2 = Q_\lambda(c) < 0 < Q_\lambda(x_1) = \lambda_1 - \lambda_2,
\]

it follows that there is a point \( y_1 \) in \( (c, x_1) \) such that \( Q_\lambda(y_1) = 0 \). But then \( Q_\lambda(x_2) = Q_\lambda(y_1) = 0 \) and \( Q_\lambda'(x_2) = 0 \), and so \( Q_\lambda(x) \) has at least three zeros (counting multiplicities) in \( I \). But by part (A), \( Q_\lambda(x) \) can have at most two zeros in \( I \). This is the desired contradiction. Thus we have established that the function \( \mathcal{D}(\varphi' / \varphi) \) has no zeros in \( I \), if \( \varphi \) vanishes on \( I \), and that \( \mathcal{D}(\varphi' / \varphi) \) has precisely one zero in \( I \), if \( \varphi(x) \neq 0 \) in \( I \).

(C) We next consider the unbounded interval \( I_\infty = (b_L, \infty) \) in \( S \), where \( b_L \) is the largest zero of \( \varphi'(x) \). Then the foregoing analysis, \textit{mutatis mutandis}, yields the following conclusions.

1. If \( \varphi \) vanishes on \( I_\infty \), then \( \mathcal{D}(\varphi' / \varphi) \neq 0 \) on \( I_\infty \).
2. If \( \varphi \neq 0 \) and \( \varphi' / \varphi > 0 \) on \( I_\infty \), then \( \mathcal{D}(\varphi' / \varphi) \) has exactly one zero, counting multiplicity, in \( I_\infty \).
3. If \( \varphi \neq 0 \) and \( \varphi' / \varphi < 0 \) on \( I_\infty \), then \( \mathcal{D}(\varphi' / \varphi) \neq 0 \) on \( I_\infty \).

(D) Finally, consider the interval \( J = (a_k, a_{k+1}) \), where \( a_k \) and \( a_{k+1} \) are two consecutive zeros of \( \varphi \), and let \( b_1 \leq b_2 \leq \cdots \leq b_{2n+1} \) be the zeros of \( \varphi' \).
in $J$. By part B (see case 2), $D(\varphi'/\varphi)$ has exactly $2n$ zeros, counting multiplicities, in $J$. That is, the number of extra zeros of $\varphi'$ on each bounded interval $J$ is equal to the number of zeros of $D(\varphi'/\varphi)$ in $J$. If $\varphi$ has a largest zero, call it $a_L$, we also need to consider the unbounded interval $J_\infty = (a_L, \infty)$. In this case, by part (C), the number of extra zeros of $\varphi'$ in $J_\infty$ is either equal to the number of zeros of $D(\varphi'/\varphi)$ in $J_\infty$ (cf. parts (C)(1) and (C)(2)) or the number of extra zeros of $\varphi'$ in $J_\infty$ is one greater than the number of real zeros of $D(\varphi'/\varphi)$ in $J_\infty$ (cf. part (C)(3)). Thus, we have proved that

$$E(\varphi') = Z_R\left(\frac{\varphi'}{\varphi}\right) \text{ or } Z_R\left(\frac{\varphi'}{\varphi}\right) + 1.$$  

We next claim that $Z_R(D(\varphi'/\varphi))$ is an even number. Indeed, we have already shown that $D(\varphi'/\varphi)$ has an even number of zeros on each of the bounded intervals $J$. Now consider $J_\infty = (a_L, \infty)$ and observe that since $\varphi \in \mathcal{L} - \mathcal{P}^*$, $D(\varphi'/\varphi) < 0$ outside a compact set (see (3.11) and the subsequent remark). Also, $D(\varphi'/\varphi)(x) < 0$ for $x$ sufficiently close to $a_L$. Hence, $D(\varphi'/\varphi)$ has an even number of sign changes in $J_\infty$ and consequently, the number of zeros of $D(\varphi'/\varphi)$ in $J_\infty$ of odd multiplicity is an even number. Therefore, $D(\varphi'/\varphi)$ has an even number of zeros both in $J$ and $J_\infty$ and so $Z_R(D(\varphi'/\varphi))$ is even. But by Lemma 3,

$$E(\varphi') + Z_c(\varphi') = 2d \quad \text{or} \quad 2d + 1.$$  

Since by assumption $Z_c(\varphi') = 0$ and since $Z_R(D(\varphi'/\varphi))$ is an even number, it follows from (3.4) and (3.5) that $Z_R(D(\varphi'/\varphi)) = 2d$.

(a) $\Rightarrow$ (b)(ii): Let $a_i$ and $b_i$, $1 \leq i \leq n$, be two consecutive zeros of $\varphi$. Suppose that $J_1 = (a_1, b_1), \ldots, J_n = (a_n, b_n)$ are the bounded intervals which contain extra zeros of $\varphi'$ and that the $a_i$'s and $b_i$'s are arranged so that $a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_n < b_n$. Since

$$\frac{\varphi'}{\varphi}(a_i +) = +\infty \quad \text{and} \quad \frac{\varphi'}{\varphi}(b_i -) = -\infty, \quad 1 \leq i \leq n,$$

$\varphi'/\varphi(x)$ has an odd number of zeros in $J_i$ and hence the number of extra zeros of $\varphi'$ in each interval $J_i$, $1 \leq i \leq n$, is even. Consider the interval $J_1 = (a_1, b_1)$ and suppose that the zeros $t_j$, $1 \leq j \leq 2m + 1$, of $\varphi'/\varphi$ in $J_1$ are simple:

$$a_1 < t_1 < t_2 < \cdots < t_{2m+1} < b_1.$$  

Since $\varphi'/\varphi(t_j) = \varphi'/\varphi(t_{j+1}) = 0$, $1 \leq j \leq 2m$, it follows from Rolle's theorem that $D(\varphi'/\varphi)$ has an odd number of zeros in each of the intervals $(t_j, t_{j+1})$, $1 \leq j \leq 2m$. Now by the argument used above (see case 2 of part (B) in the proof of the implication (a) $\Rightarrow$ (b)(i)), we know that $D(\varphi'/\varphi)$ has exactly one
zero, call it $x_j$, in the interval $(t_j, t_{j+1})$, $1 \leq j \leq 2m$. Thus, $a_1 < t_1 < x_1 < t_2 < \cdots < x_{2m} < t_{2m+1} < b_1$ and using (3.6) we see that
\[
D\left(\frac{\varphi'}{\varphi}\right) < 0 \quad \text{for} \quad x \in (a_1, x_1) \cup (x_2, x_3) \cup \cdots \cup (x_{2m}, b)
\]
and
\[
D\left(\frac{\varphi'}{\varphi}\right) > 0 \quad \text{for} \quad x \in (x_1, x_2) \cup (x_3, x_4) \cup \cdots \cup (x_{2m-1}, x_{2m}).
\]
Therefore, it follows that
\[
\frac{\varphi'}{\varphi}(x_{2j-1}) < 0 \quad \text{and} \quad \frac{\varphi'}{\varphi}(x_{2j}) > 0, \quad 1 \leq j \leq m.
\]
Moreover, it is clear that $\varphi(x) \neq 0$ for $x_{2j-1} \leq x \leq x_{2j}$, $1 \leq j \leq m$. If the zeros $t_j$ of $\varphi'/\varphi$ in $I_1$ are not all distinct, then the foregoing argument, in conjunction with multiplicity considerations, yields that
\[
(3.7) \quad \frac{\varphi'}{\varphi}(x_{2j-1}) \leq 0, \quad \frac{\varphi'}{\varphi}(x_{2j}) \geq 0
\]
and
\[
(3.8) \quad \varphi(x) \neq 0 \quad \text{for} \quad x_{2j-1} \leq x \leq x_{2j}, \quad 1 \leq j \leq m.
\]
Since the number of zeros of $D(\varphi'/\varphi)$ in $I_i$, $1 \leq i \leq n$, is even, it is clear that the conclusions (3.7) and (3.8) remain valid when the zeros $x_j$ of $D(\varphi'/\varphi)$ lie in the other bounded intervals $I_2, \ldots, I_n$.

Finally, we note that if $\varphi$ has a largest zero, say $a_L$, then the number of extra zeros of $\varphi'/\varphi$ in $I_\infty = (a_L, \infty)$ may be even or odd. In either case, we know that the number of zeros of $D(\varphi'/\varphi)$ in $I_\infty$ is even (cf. the proof of the implication (a) $\Rightarrow$ (b)(i)). Thus, the foregoing analysis shows, mutatis mutandis, the (3.7) and (3.8) remain true when the zeros $x_j$ of $D(\varphi'/\varphi)$ lie in the unbounded interval $I_\infty = (a_L, \infty)$.

II. Next consider the case when $\varphi(x)$ has only a finite number of real zeros. First, suppose that $\varphi(x)$ has at least one real zero, so that there are two unbounded intervals of the form $(-\infty, a_i)$ and $(a_L, \infty)$, where $a_i$ is the smallest zero of $\varphi$ and $a_L$ is the largest zero of $\varphi$. Then, degree considerations show that the number of extra zeros of $\varphi'$ on at least one of these unbounded intervals, say $(a_L, \infty)$, must coincide with the number of real zeros of $D(\varphi'/\varphi)$ on $(a_L, \infty)$. Therefore, the argument used in part I is applicable in this case as well.

In conclusion, suppose that $\varphi(x) = e^{ax}p(x)$, $\alpha \in \mathbb{R}$, where $p(x)$ is a real polynomial of degree $2d$ with no real zeros. The proof in this case is similar to
the proof in part I, except that we apply Remark 3.1(b) in place of Lemma 3, if $\alpha = 0$. \qed

**Remark 3.3.** We remark that Theorem 1 remains valid if $\varphi \in \mathcal{L} - \mathcal{P}^*$, $Z_\alpha(\varphi) = 2d$ and the order, $\rho(\varphi)$, is 2. This remark also applies to Corollary 1 and Lemma 4.

**Remark 3.4.** In the proof of the implication (b) $\Rightarrow$ (a) of Theorem 1, we have only used the inequality $Z_R(D(\varphi'/\varphi)) \geq 2m$, where $m \geq d$, to conclude that $E(\varphi'/\varphi) \geq 2d$. But then the implication (a) $\Rightarrow$ (b) of Theorem 1, with $\gamma = 0$, shows that, in fact, $Z_R(D(\varphi'/\varphi)) = 2d$. Moreover, it follows from Lemma 3 that $0 \leq E(\varphi'/\varphi) - Z_R(D(\varphi'/\varphi)) \leq 1$.

Now suppose that $\varphi \in \mathcal{L} - \mathcal{P}^*$, $\varphi$ has exactly $2d$ nonreal zeros, $(D + \gamma)\varphi(x) \in \mathcal{L} - \mathcal{P}$ for all $\gamma$ in an open interval $I$ and that $\gamma_1, \gamma_2, \gamma_1 < \gamma_2$, satisfy $[\gamma_1, \gamma_2] \subseteq I$ and $0 \notin [\gamma_1, \gamma_2]$. Then by Theorem 1, $\varphi'/\varphi(x)$ has $2d$ local extrema on some interval $[a, b]$, with $d$ local maxima strictly greater than $-\gamma_1$ and $d$ local minima strictly less than $-\gamma_2$. Consider the rectangle $R = \{(x, y)|a \leq x \leq b, -\gamma_2 \leq y \leq -\gamma_1\}$. If $m$ is sufficiently large, then, for any $\gamma_0$ satisfying $\gamma_1 < \gamma_0 < \gamma_2$, the portion of the hyperbola $y = (-m)/(x + m/\gamma_0), a \leq x \leq b$, lies entirely in the rectangle $R$. It follows that the function $(x + m/\gamma_0)^m\varphi(x)$ satisfies statement (b) of Theorem 1 (with $\gamma = 0$), and so $D(x + m/\gamma_0)^m\varphi(x) \in \mathcal{L} - \mathcal{P}$. Thus we have established the following lemma.

**Lemma 4.** Let $\varphi \in \mathcal{L} - \mathcal{P}^*$. Suppose that $\rho(\varphi) < 2$ and that $\varphi$ has exactly $2d$, $d \geq 0$, nonreal zeros. Suppose $(D + \gamma)\varphi(x) \in \mathcal{L} - \mathcal{P}$ for all $\gamma$ in an open nonempty interval $I$. Then for any interval $[\gamma_1, \gamma_2] \subseteq I$, $0 \notin [\gamma_1, \gamma_2]$, there is a positive integer $m_0$ such that

$$D(x + m/\gamma)^m\varphi(x) \in \mathcal{L} - \mathcal{P}, \quad \text{for all } \gamma \in [\gamma_1, \gamma_2]$$

and for all $m \geq m_0$.

The proposition to be proved below will require the following two additional lemmas concerning functions in $\mathcal{L} - \mathcal{P}^*$.

**Lemma 5.** Let $\varphi \in \mathcal{L} - \mathcal{P}^*$.

(a) If $(D + \gamma)\varphi \in \mathcal{L} - \mathcal{P}$ for some $\gamma \in \mathbb{R}$, then

$$(D + \gamma)^k\varphi(x) \in \mathcal{L} - \mathcal{P}, \quad k \geq 1.$$

(b) If $D(x + a)^m\varphi(x) \in \mathcal{L} - \mathcal{P}$, where $a \in \mathbb{R}$ and $m$ is a positive integer, then

$$D(x + a)^{m+k}\varphi(x) \in \mathcal{L} - \mathcal{P}, \quad k \geq 1.$$
Proof. (a) Since \((D + \gamma)\varphi \in \mathcal{L} - \mathcal{P}\), we obtain
\[ D^k(D + \gamma)\varphi = (D + \gamma)D^k\varphi \in \mathcal{L} - \mathcal{P}. \]

(b) Since \(D(x + a)^m \varphi = (x + a)^{m-1}[m \varphi + (x + a)\varphi'] \in \mathcal{L} - \mathcal{P}\), it follows that \(m \varphi + (x + a)\varphi' \in \mathcal{L} - \mathcal{P}\). But then
\[ D^k[(x + a)\varphi' + m \varphi] = (m + k)\varphi^{(k)}(x) + (x + a)\varphi^{(k+1)}(x) \in \mathcal{L} - \mathcal{P}. \]
On the other hand,
\[ D\left[(x + a)^{m+k} \varphi^{(k)}(x) \right] = (x + a)^{m+k-1}[(m + k)\varphi^{(k)}(x) + (x + a)\varphi^{(k+1)}(x) - \dots, \]
and the result follows.

**Lemma 6.** Let \(\varphi(x) \in \mathcal{L} - \mathcal{P}^*\) and suppose that \(Z_{c}(\varphi^{(k)}(x)) = 2d > 0, \) for all integers \(k \geq 0\). Suppose also that for any \(k \geq 0\) there are \(2d\) real numbers \(x_{j,k}\) such that
\[ x_{1,k} < x_{2,k} < \cdots < x_{2d,k}, \]
and
\[ D \frac{\varphi^{(k+1)}}{\varphi^{(k)}}(x_{j,k}) = 0, \quad 1 \leq j \leq 2d, \quad k \geq 0. \]
Then
\[
(3.10) \quad x_{j,k} = O(\sqrt{k}), \quad 1 \leq j \leq 2d, \quad \text{as } k \rightarrow \infty.
\]

**Proof.** To prove (3.10), let
\[ \varphi^{(k)}(x) = \left( \prod_{j=1}^{d} \left[ (x - \alpha_{j,k})^2 + \beta_{j,k}^2 \right] \right) \psi_k(x), \quad k \geq 0, \]
where \(\psi_k(x) \in \mathcal{L} - \mathcal{P}\) and \(\beta_{j,k} > 0\). Then a calculation yields that
\[
(3.11) \quad D \frac{\varphi^{(k+1)}}{\varphi^{(k)}}(x) = 2 \sum_{i=1}^{d} \frac{\beta_{i,k}^2 - (x - \alpha_{i,k})^2}{\beta_{i,k}^2 + (x - \alpha_{i,k})^2} + D \frac{\psi'_k}{\psi_k}(x).
\]
Since \(D(\varphi^{(k+1)}/\varphi^{(k)})(x_{j,k}) \geq 0, 1 \leq j \leq 2d, \ k \geq 0, \) and since
\[ D(\psi'_k/\psi_k)(x) \leq 0, \]
it follows from (3.11) that for some integer \(i, 1 \leq i \leq d, \)
\[
(3.12) \quad \alpha_{i,k} - \beta_{i,k} \leq x_{j,k} \leq \alpha_{i,k} + \beta_{i,k}.
\]
But by the Jensen-Nagy-Walsh theorem
\[ |\alpha_{j,k} + i\beta_{j,k}| = O(\sqrt{k}), \quad 1 \leq j \leq d, \quad \text{as } k \rightarrow \infty. \]
The desired growth estimate (3.10) is a consequence of (3.12) and (3.13).
We pause for a moment to motivate the statement and the proof of the following proposition. In this proposition we will assume that \( Z_{\gamma}(\varphi^{(k)}) = 2d \) for all nonnegative integers \( k \) and that \((D + \gamma)\varphi(x) \in L - P\) for all \( \gamma \) in an open, nonempty interval \( I \). By virtue of these assumptions we will be able to estimate the imaginary parts of the \( 2d \) nonreal zeros of \( \varphi^{(k)}(x) \). These estimates, in turn, will be used to prove Theorem 2 by contradiction.

The basic intuitive and geometric ideas underlying the proof of this proposition may be sketched as follows. By the foregoing results (cf. Theorem 1, Lemma 4 and Lemma 5) the graph of \( y = (\varphi^{(k+1)}/\varphi^{(k)})(x), \ k \geq 0, \) has \( d \) local minima which lie below the line \( y = -\gamma_0 \) and has \( d \) local maxima which lie above the line \( y = -\gamma_0 \), where \( \gamma_0 \in I \) and, for reasons of convenience, it is assumed that \( \gamma_0 > 0 \). If we label the abscissae of these \( 2d \) local extrema by \( x_{1,k} < \cdots < x_{2d,k} \), then the graph of \( y = \varphi^{(k+1)}/\varphi^{(k)}(x) \) is increasing on each of the \( d \) intervals \( J_{j,k} = [x_{2j-1,k}, x_{2j,k}], \ 1 \leq j \leq d. \) Moreover, \( \varphi^{(k+1)}/\varphi^{(k)}(x) < 0 \) on each interval \( J_{j,k}, 1 \leq j \leq d. \) Then, by way of a technical argument, we will prove that for a certain positive integer \( m_0 \) and for all sufficiently large positive integers \( k \), the branch of the hyperbola \( y = -(m_0 + k)/(x + a), \ a = m_0/\gamma_0, \) which lies to the right of the vertical asymptote \( x = -a \), intersects the graph of \( y = \varphi^{(k+1)}/\varphi^{(k)}(x) \) on each interval \( J_{j,k}, 1 \leq j \leq d. \) But then with increasing \( k \), this branch of the hyperbola, when restricted to the set \( J_k = \bigcup_{j=1}^{d} J_{j,k} \), moves away from the line \( y = -\gamma_0 \). Consequently, the vertical separation between a local minimum and the succeeding local maximum, affected by the imaginary parts \( \pm \beta_{j,k}, \ \beta_{j,k} > 0, \ 1 \leq j \leq d, \) of the nonreal zeros of \( \varphi^{(k)}(x), \) has order of magnitude \((m_0 + k)/(x_{2j,k} + a), \) as \( k \to \infty. \) From this it will then follow that \( \beta_{j,k} \leq c(x_{2j,k} + a)^{-1}, \ 1 \leq j \leq d, \ k \geq 1. \) These estimates, together with the Jensen-Nagy-Walsh theorem, will be used to prove Theorem 2.

**Proposition.** Let \( \varphi(x) \in L - P^*, \ \rho(\varphi) < 2, \) and suppose that \( Z_{\gamma}(\varphi^{(k)}(x)) = 2d > 0, \) for all integers \( k \geq 0. \) Let

\[
\varphi^{(k)}(\alpha_{j,k} \pm i\beta_{j,k}) = 0, \quad \beta_{j,k} > 0, \quad 1 \leq j \leq d, \quad k \geq 0.
\]

Further, suppose that \((D + \gamma)\varphi(x) \in L - P\) for all \( \gamma \) in an open nonempty interval \( I. \) Then for any \( \gamma_0 \) in \( I \) there is a positive integer \( m_0, \) depending only on \( \gamma_0 \) and \( \varphi, \) such that

\[
\beta_{j,k} \leq c_1 A_k/k, \quad 1 \leq j \leq d, \quad k \geq 1,
\]

where

\[
A_k = \max\left(1, \max_{1 \leq j \leq d} |\alpha_{j,k} + a|\right), \quad \text{and} \quad a = \frac{m_0}{\gamma_0},
\]

and where \( c_1 \) is a constant independent of \( k. \)
Proof. First we observe that the assumptions that $Z_c(\varphi^{(k)}(x)) = 2d > 0$, for all $k \geq 0$, and that $(D + \gamma)\varphi(x) \in \mathcal{L} - \mathcal{P}$, imply that the point $\gamma = 0$ is not in the interval $I$. Let $[\gamma_1, \gamma_2]$, $\gamma_1 < \gamma_2$, be any closed interval in $I$. Then we may assume, without loss of generality, that $0 < \gamma_1 < \gamma_2$. Since $(D + \gamma)\varphi(x) \in \mathcal{L} - \mathcal{P}$, for all $\gamma \in [\gamma_1, \gamma_2]$, it follows from Lemma 5 that $(D + \gamma)\varphi^{(k)}(x) \in \mathcal{L} - \mathcal{P}$, for all $\gamma \in [\gamma_1, \gamma_2]$ and $k \geq 0$. Hence by Theorem 1 and Corollary 1, for any $\gamma \in [\gamma_1, \gamma_2]$ and for any $k \geq 0$, there are exactly $2d$ real numbers $x_{j,k}$ such that

$$
(3.17) \quad x_{1,k} < x_{2,k} < \cdots < x_{2d,k}, \\
D \frac{\varphi^{(k+1)}}{\varphi^{(k)}}(x_{j,k}) = 0, \quad 1 \leq j \leq 2d, \quad k \geq 0,
$$

and such that

$$
(3.18) \quad \frac{\varphi^{(k+1)}}{\varphi^{(k)}}(x_{2j-1,k}) < -\gamma \quad \text{and} \quad \frac{\varphi^{(k+1)}}{\varphi^{(k)}}(x_{2j,k}) > -\gamma,
$$

$$
1 \leq j \leq d, \quad k \geq 0,
$$

and such that

$$
\varphi^{(k)}(x) \neq 0 \quad \text{for} \quad x_{2j-1,k} \leq x \leq x_{2j,k}, \quad 1 \leq j \leq d, \quad k \geq 0.
$$

Moreover, since $x_{2j-1,k}$ and $x_{2j,k}$ are consecutive zeros of $D(\varphi^{(k+1)}/\varphi^{(k)})(x)$, and since $\varphi \in \mathcal{L} - \mathcal{P}^*$, it follows from (3.18) that

$$
(3.19) \quad D \frac{\varphi^{(k+1)}}{\varphi^{(k)}}(x) \geq 0 \quad \text{if and only if} \quad x \in J_k,
$$

where

$$
(3.20) \quad J_k = \bigcup_{j=1}^{d} J_{j,k} \quad \text{and} \quad J_{j,k} = [x_{2j-1,k}, x_{2j,k}], \quad 1 \leq j \leq d, \quad k \geq 0.
$$

Now by assumption $Z_c(\varphi^{(k)}(x)) = 2d > 0$, $k \geq 0$, and so by Lemma 3, $E(\varphi^{(k+1)}/\varphi^{(k)}(x)) = 0$ or 1. But then it follows that

$$
(3.21) \quad \frac{\varphi^{(k+1)}}{\varphi^{(k)}}(x) < 0 \quad \text{for all} \quad x \in J_k, \quad k \geq 0.
$$

Indeed, the inequality $\varphi^{(k+1)}/\varphi^{(k)}(x) \geq 0$ for some $x \in J_k$ and for some $k \geq 0$ would imply that $E(\varphi^{(k+1)}/\varphi^{(k)}(x)) \geq 2$ and this, in turn, would contradict our conclusion that $E(\varphi^{(k+1)}/\varphi^{(k)}(x)) = 0$ or 1.

Next, since $(D + \gamma)\varphi(x) \in \mathcal{L} - \mathcal{P}$, for all $\gamma \in [\gamma_1, \gamma_2]$, we can invoke Lemma 4 to conclude that there is a positive integer $m_1$ such that $D(x + m/\gamma)^m \varphi(x) \in \mathcal{L} - \mathcal{P}$, for all $\gamma \in [\gamma_1, \gamma_2]$ and for all $m \geq m_1$. But then by Lemma 5

$$
(3.22) \quad D(x + m/\gamma)^{m+k} \varphi^{(k)}(x) \in \mathcal{L} - \mathcal{P}, \quad \text{for all} \quad \gamma \in [\gamma_1, \gamma_2]
$$

and for all $m \geq m_1$, $k \geq 0$. 

Now for a fixed $\gamma_0$ in $(\gamma_1, \gamma_2)$ and for a fixed positive integer $m$, $m \geq m_1$, set
\[ T_{k, m}(x) = T_{k, m}(x; \gamma_0) = \frac{m + k}{x + m/\gamma_0} + \frac{\varphi^{(k+1)}(x)}{\varphi^{(k)}}(x), \quad k \geq 0. \]

Then by (3.22) and Theorem 1 there are $2d$ real numbers, $y_{1,k} \leq y_{2,k} \leq \cdots \leq y_{2d,k}$, $k \geq 0$, such that
\[
(3.23) \quad T_{k, m}(y_{j,k}) = 0, \quad 1 \leq j \leq 2d, \quad k \geq 0,
\]
\[
(3.24) \quad T_{k, m}(y_{2j-1,k}) \leq 0 \quad \text{and} \quad T_{k, m}(y_{2j,k}) \geq 0, \quad 1 \leq j \leq d, \quad k \geq 0.
\]
Since $D((m + k)/(x + m/\gamma)) < 0$, (3.23) implies that
\[
D\frac{\varphi^{(k+1)}}{\varphi^{(k)}}(y_{j,k}) > 0, \quad 1 \leq j \leq 2d, \quad k \geq 0,
\]
and so by (3.19), $y_{j,k} \in J_k$ for $1 \leq j \leq 2d$ and $k \geq 0$.

Next, our proof of the estimates (3.15) will be based on the following two assertions. There is a positive integer $m_0$, where $m_0$ is independent of $k$, and there is a positive integer $k_0$ such that for each $k \geq k_0$, there are $d$ real numbers $t_{j,k} \in J_{j,k}$, $1 \leq j \leq d$, such that
\[
(3.25) \quad T_{k, m_0}(t_{j,k}) = T_{k, m_0}(t_{j,k}; \gamma_0) = 0.
\]
Moreover, there is a positive integer $k_1$, $k_1 \geq k_0$, such that for $k \geq k_1$,
\[
(3.26) \quad \text{each interval } J_{j,k} \text{ contains exactly two zeros of } T_{k, m_0}(x).
\]

For the sake of clarity of presentation, the somewhat lengthy proofs of assertions (3.25) and (3.26) will be deferred to Lemma 7 and Lemma 8, which are given below.

Thus by (3.26) the zeros $y_{j,k}$ of $T_{k, m_0}(x)$, $k \geq k_1$, satisfy the inequalities
\[
(3.27) \quad x_{2j-1,k} < y_{2j-1,k} \leq y_{2j,k} < x_{2j,k}, \quad 1 \leq j \leq d.
\]
We next observe that for $k \geq k_1$,
\[
(3.28) \quad y_{j,k} + \frac{m_0}{\gamma_0} > 0, \quad 1 \leq j \leq 2d.
\]
Indeed, it follows from Lemma 8 that (3.28) holds for $1 \leq j \leq 2$ (cf. (3.43)), while for $3 \leq j \leq 2d$ inequality (3.28) follows from (3.17), (3.40) and (3.27). Hence if we set
\[
\Gamma_k = \frac{m_0 + k}{y_{2d-1,k} + m_0/\gamma_0}, \quad k \geq k_1,
\]
then for $1 \leq j \leq d$ we have

$$
\begin{align*}
\frac{\varphi^{(k+1)}}{\varphi^{(k)}}(x_{2j-1,k}) &< \frac{\varphi^{(k+1)}}{\varphi^{(k)}}(y_{2j-1,k}) \\
&\leq -\frac{m_0 + k}{y_{2j-1,k} + m_0/\gamma_0} \\
&\leq -\Gamma_k.
\end{align*}
$$

(3.29)

Since by Lemma 6, $x_{j,k} = O(\sqrt{k})$, $1 \leq j \leq 2d$, as $k \to \infty$, there is an integer $k_2 \geq k_1$, such that $\Gamma_k > 2\gamma_0$ for all $k \geq k_2$. For a fixed $k$, $k \geq k_2$, let $p_i(x) = (x - \alpha_{i,k})^2 + \beta_{i,k}^2$, $1 \leq i \leq d$, and set $f_i(x) = \varphi^{(k)}/p_i(x)$. Then

$$
\begin{align*}
\frac{\varphi^{(k+1)}}{\varphi^{(k)}}(x) &= \frac{f_i'}{f_i}(x) + \frac{p_i'}{p_i}(x),
\end{align*}
$$

(3.30)

and $Z_c(f_i) = 2d - 2$. Thus, by Lemma 3, $E((D + r)f_i) \leq 2d - 1$ for all $r \in \mathbb{R}$.

Next suppose, for the sake of argument, that

$$
\left| \frac{p_i'}{p_i}(x) \right| \leq \frac{\Gamma_k - \gamma_0}{2} \quad \text{for all } x \in \mathbb{R}.
$$

But then by (3.18), (3.29) and (3.30),

$$
\frac{f_i'}{f_i}(x_{2j-1,k}) < -\frac{\Gamma_k + \gamma_0}{2} \quad \text{and} \quad \frac{f_i'}{f_i}(x_{2j,k}) > -\frac{\Gamma_k + \gamma_0}{2}.
$$

By counting, as in the proof that (b) implies (a) in Theorem 1, we conclude that

$$
E\left(\left( D + \frac{\Gamma_k + \gamma_0}{2} \right)f_i \right) \geq 2d.
$$

This is the desired contradiction, since $E((D + r)f_i) \leq 2d - 1$ for all $r \in \mathbb{R}$.

Therefore, for some $x \in \mathbb{R}$,

$$
\left| \frac{p_i'}{p_i}(x) \right| > \frac{\Gamma_k - \gamma_0}{2},
$$

and hence

$$
\sup\left\{ \left| \frac{p_i'}{p_i}(x) \right| : x \in \mathbb{R} \right\} = \frac{1}{\beta_{i,k}} > \frac{\Gamma_k - \gamma_0}{2}, \quad 1 \leq i \leq d.
$$

Thus for $k \geq k_2$ and $1 \leq j \leq d$, with $a = m_0/\gamma_0$,

$$
\beta_{j,k} < \frac{2}{\Gamma_k - \gamma_0} < \frac{4}{\Gamma_k} = 4 \frac{y_{2d-1,k} + a}{m_0 + k} < \frac{4}{k}.
$$

(3.31)
Since $D(q^{(k+1)}/q^{(k)})(y_{2d-1,k}) > 0$, the argument following (3.11) shows that there is an integer $i$, $1 \leq i \leq d$, such that
\begin{equation}
\alpha_{i,k} - \beta_{i,k} < y_{2d-1,k} < \alpha_{i,k} + \beta_{i,k}.
\end{equation}
Also, by the Jensen-Nagy-Walsh theorem,
\begin{equation}
\beta_{i,k} \leq \max_{1 \leq j \leq d} \{ \beta_{j,0} \} = c.
\end{equation}
Then by (3.31), (3.32) and (3.33), for $1 \leq j \leq d$, $k \geq k_2$,
\begin{equation}
\beta_{j,k} < 4 \frac{y_{2d-1,k} + a}{k} < 4 \frac{\alpha_{i,k} + a + \beta_{i,k}}{k}
\end{equation}
\begin{equation}
< 4 \frac{A_k + c}{k} \leq 4(1 + c) \frac{A_k}{k},
\end{equation}

since $A_k$, defined by (3.16), is at least one. The desired estimate (3.15) now follows and the proof of the proposition is complete.

In the following two lemmas we assume that the hypotheses of the proposition hold. Moreover, we use the notation adopted in the proof of the proposition.

**Lemma 7.** With the hypotheses of the proposition, assertion (3.25) is valid.

**Proof.** To prove (3.25), we choose $\varepsilon > 0$ such that $(\gamma_0 - 2\varepsilon, \gamma_0 + 2\varepsilon) \subseteq [\gamma_1, \gamma_2]$. Next, let $m_0$ be a positive integer, $m_0 \geq m_1$, such that
\begin{equation}
\frac{m}{\gamma_0 - \varepsilon} \geq \frac{m + 1}{\gamma_0}
\end{equation}
for all $m \geq m_0$.

Moreover, by (3.10) there is a positive integer $k_0$ such that for all $k \geq k_0$,
\begin{equation}
k \geq \gamma_0 x_{2d,k}.
\end{equation}
Now suppose that there is an integer $k$, $k \geq k_0$, and an integer $j$, $1 \leq j \leq d$, such that
\begin{equation}
T_{k,m_0}(x) = T_{k,m_0}(x; \gamma_0) = \frac{m_0 + k}{x + m_0/\gamma_0} + \frac{q^{(k+1)}}{q^{(k)}}(x) \neq 0
\end{equation}
for all $x \in I_{j,k}$. Since $\gamma_0 + q^{(k+1)}/q^{(k)}(x)$ has a simple zero in $I_{j,k}$ (cf. (3.18)) and since $(m + k)/(x + m/\gamma_0) \to \gamma_0$ uniformly on compact subsets of $C$, as $m \to \infty$, we can find a positive integer $m_2 \geq m_0$ such that
\begin{equation}
T_{k,m_2}(x; \gamma_0) \neq 0
\end{equation}
for all $x \in I_{j,k}$

and such that
\begin{equation}
T_{k,m_2+1}(t_0; \gamma_0) = 0
\end{equation}
for some $t_0 \in I_{j,k}$. 

Thus by (3.36) and (3.21)

$$\frac{m_2 + 1 + k}{t_0 + (m_2 + 1)/\gamma_0} = \frac{\phi^{(k+1)}}{\phi^{(k)}}(t_0) < 0,$$

and consequently $t_0 + (m_2 + 1)/\gamma_0 > 0$. Thus, by (3.34),

$$t_0 + \frac{m_2}{\gamma_0 - \epsilon} \geq t_0 + \frac{m_2 + 1}{\gamma_0} > 0.$$

Since $t_0 \in J_{j,k}$, $t_0 \leq x_{2d,k}$ and so by (3.35), for $k \geq k_0$, $k \geq \gamma_0 t_0$ and for $u \geq m_2$,

$$\frac{d}{du} \left( \frac{u + k}{t_0 + u/\gamma_0} \right) = \left( t_0 + \frac{u}{\gamma_0} \right)^{-2} \left( t_0 - \frac{k}{\gamma_0} \right) \leq 0.$$

Thus we have for $k \geq k_0$,

$$\begin{align*}
\frac{m_2 + k}{t_0 + m_2/(\gamma_0 - \epsilon)} &> -\frac{(m_2 + 1) + k}{t_0 + m_2/(\gamma_0 - \epsilon)} \\
&\geq -\frac{(m_2 + 1) + k}{t_0 + (m_2 + 1)/\gamma_0} \\
&= \frac{\phi^{(k+1)}}{\phi^{(k)}}(t_0) \\
&> -\frac{m_2 + k}{t_0 + m_2/\gamma_0}.
\end{align*}$$

(3.38)

Now consider

$$T_{k,m_2}(x; s) = \frac{m_2 + k}{x + m_2/s} + \frac{\phi^{(k+1)}}{\phi^{(k)}}(x), \quad s > 0, \quad k \geq k_0,$$

and let

$$S = \{ s : s > \gamma_0 - \epsilon \text{ and } T_{k,m_2}(x; s) \text{ has a zero in } J_{j,k} \}$$

and $\hat{\gamma} = \inf S$. The last inequality of (3.38) implies that $t_0 + m_2/\gamma_0 > 0$ and so it follows that $T_{k,m_2}(t_0; s)$ is continuous for $\gamma_0 - \epsilon \leq s \leq \gamma_0$. Also, by (3.38)

$$T_{k,m_2}(t_0; \gamma_0 - \epsilon) < 0 \quad \text{and} \quad T_{k,m_2}(t_0; \gamma_0) > 0,$$

and consequently $T_{k,m_2}(t_0; s) = 0$ for some $s$, $\gamma_0 - \epsilon < s < \gamma_0$. Thus, $S \neq \emptyset$ and $\gamma_0 - \epsilon < \hat{\gamma} < \gamma_0$. Moreover, by the definition of $\hat{\gamma}$ it is easy to see that $\hat{\gamma} \in S$ and that $T_{k,m_2}(x; \hat{\gamma})$ has a zero, call it $\hat{\gamma}$, in $J_{j,k}$, of order at least two. Set $B(\hat{\gamma}, \delta) = \{ z : |z - \hat{\gamma}| < \delta \}$, where $0 < \delta < \min\{|\gamma_0 - \epsilon - \hat{\gamma}|, |\gamma_0 - \hat{\gamma}|\}$, and
let \((\lambda_n)_{n=1}^{\infty}\) be a sequence of real numbers such that \(\gamma_0 - \epsilon < \lambda_n < \hat{\gamma} < \lambda_n + 1 > \lambda_n\) and \(\lim_{n \to \infty} \lambda_n = \hat{\gamma}\). Since \(\lambda_n < \hat{\gamma}\), \(T_{k, m_0}(x; \lambda_n) \neq 0\) on \(J_{j,k}\) and hence the function

\[ h_n(x) = D(x + m_0/\lambda_n)^{m_2 + k} \varphi^{(k)}(x) \neq 0 \quad \text{on} \quad J_{j,k}. \]  

But \(h_n(x) \to D(x + m_2/\hat{\gamma})^{m_2 + k} \varphi^{(k)}(x) = \hat{h}(x)\), uniformly on \(\overline{B(\hat{\gamma}, \delta)}\), as \(n \to \infty\). Hence, by Hurwitz's theorem there is a positive integer \(n_0\) such that \(h_n(x), n \geq n_0\), and \(\hat{h}(x)\) have the same number of zeros, counting multiplicities, in \(B(\hat{\gamma}, \delta)\). By (3.39) the zeros of \(h_n(x), n \geq n_0\), in \(B(\hat{\gamma}, \delta)\) are all nonreal. Since \(\hat{h}\) has at least two zeros in \(B(\hat{\gamma}, \delta)\) and \(\lambda_n \in (\gamma_0 - \epsilon, \gamma_0) \subseteq [\gamma_1, \gamma_2]\), this contradicts (3.22) and thus we have proved assertion (3.25).

**Lemma 8.** With the hypotheses of the proposition, assertion (3.26) is valid.

**Proof.** Since \(T'_{k, m_0}(x)\) has exactly \(2d\) real zeros (cf. (3.23)), it suffices to show that there is a positive integer \(k_1 > k_0\) such that each interval \(J_{j,k}\), \(1 \leq j \leq d, k \geq k_1\), contains at least two zeros of \(T'_{k, m_0}(x)\). To this end we first observe that, since \(T_{k, m_0}(x)\) vanishes on each interval \(J_{j,k}, k \geq k_0\),

\[ x + m_0/\gamma_0 > 0 \quad \text{whenever} \quad x \geq x_{2,k}, \quad k \geq k_0. \]  

Indeed, if \(x + m_0/\gamma_0 \leq 0\) for some \(x \geq x_{2,k}, k \geq k_0\), then \(x + m_0/\gamma_0 < 0\) for \(x \in [x_{1,k}, x_{2,k}]\). But \((\varphi^{(k+1)}(\varphi^{(k)})(x) < 0 \text{ for all } x \in \bigcup_{j=1}^{d} J_{j,k}\) (cf. (3.21)), and hence

\[ T_{k, m_0}(x) = \frac{m_0 + k}{x + m_0/\gamma_0} + \frac{\varphi^{(k+1)}}{\varphi^{(k)}}(x) < 0 \]

for \(x \in J_{1,k}, k \geq k_0\). This contradicts assertion (3.25), and thus (3.40) holds.

Next by (3.10) and (3.40) there is a positive integer \(k_1, k_1 \geq k_0\) such that

\[ \frac{m_0 + k}{x_{2,j,k} + m_0/\gamma_0} > \gamma_0 \quad \text{for all } k \geq k_1, \quad 1 \leq j \leq d. \]  

Now fix an interval \(J_{j,k} = [x_{2j-1,k}, x_{2j,k}], k \geq k_1, 1 \leq j \leq d\). Then by assertion (3.25), \(T_{k, m_0}\) has a largest zero in \(J_{j,k}\); call it \(\bar{x}\). By (3.41) and (3.18)

\[ \frac{m_0 + k}{x_{2,j,k} + m_0/\gamma_0} > \gamma_0 > -\frac{\varphi^{(k+1)}}{\varphi^{(k)}}(x_{2,j,k}), \]

and hence

\[ T_{k, m_0}(x) = \frac{m_0 + k}{x + m_0/\gamma_0} + \frac{\varphi^{(k+1)}}{\varphi^{(k)}}(x) > 0 \quad \text{for } x \in (\bar{x}, x_{2,j,k}]. \]
But then $T_{k,m_0}(\bar{x}) \geq 0$. By (3.40) $x_{j,k} + m_0/\gamma_0 > 0$ for $2 \leq j \leq 2d$ and so $T'_{k,m_0}(x)$ is continuous on $J_{j,k}$, $2 \leq j \leq d$. Since

$$T'_{k,m_0}(x_{2j-1,k}) < 0 \quad \text{and} \quad T'_{k,m_0}(x_{2j,k}) < 0,$$

it follows that $T'_{k,m_0}(x)$ has at least two zeros in $J_{j,k}$, $2 \leq j \leq d$. The same considerations apply for $J_{1,k}$, if $-m_0/\gamma_0 \leq x_{1,k}$. If $x_{1,k} < -m_0/\gamma_0 < x_{2,k}$, then $T_{k,m_0}(x) < 0$ on $[x_{1,k}, -m_0/\gamma_0]$. So the largest zero $\bar{x}$ of $T_{k,m_0}(x)$ in $J_{1,k}$ satisfies the inequality $\bar{x} > -m_0/\gamma_0$. Also, $T'_{k,m_0}(-m_0/\gamma_0 + \delta) < 0$ for all positive $\delta$ sufficiently small and $T'_{k,m_0}(x_{2,k}) < 0$. Since $T'_{k,m_0}(\bar{x}) \geq 0$, it follows that $T'_{k,m_0}(x)$ has at least two zeros, say $y_{1,k}$ and $y_{2,k}$ in $(-m_0/\gamma_0, x_{2,k}) \subseteq J_{1,k}$, which satisfy

$$y_{j,k} + \frac{m_0}{\gamma_0} > 0 \quad \text{for} \quad j = 1, 2.$$  \hfill (3.43)

This completes the proof of assertion (3.26).  \hfill \Box

In the proof of Theorem 2 we will first show that the sequence $\{A_{2^k}\}_{k=0}^{\infty}$ is bounded, where $A_k$ is defined by (3.16) of the proposition. We will use the Jensen-Nagy-Walsh theorem and the proposition to estimate the nonreal zeros of the $2^{k+1}$-st derivative of $\varphi$ in terms of the nonreal zeros of the $2^k$-th derivative of $\varphi$. From this it will follow that the Jensen ellipses corresponding to the nonreal zeros of $\varphi^{(n)}(x)$ converge to points on the real axis as $n \to \infty$. It will be shown that there is a constant $c$ such that if $y_1$ is a limit point of the Jensen ellipses, then the disk with center $y_1$ and radius $cn^{-1/2}$ will contain at least one nonreal zero of $\varphi^{(n)}(x)$. Finally, we will use these estimates, in conjunction with the following classical result of Ålander [A1] (see also [P4]), to complete the proof of Theorem 2.

**Theorem (Ålander).** Let $f(z)$ be an entire function of finite order $\rho = \rho(f)$. Let $\lambda > \rho$ and let $w \in \mathbb{C}$. Then there are infinitely many positive integers $n$ such that if $f^{(n)}(z_n) = 0$, then $|z_n - w| > (\log 2)n^{-1+1/\lambda}$.

**Theorem 2.** Let $\varphi(x) \in \mathcal{L} - \mathcal{P}^*$ and suppose that $\rho(\varphi) < 2$. If $(D + \gamma)\varphi(x) \in \mathcal{L} - \mathcal{P}$ for all $\gamma$ in an open nonempty interval $I$, then there is a positive integer $m$ such that $D^m\varphi(x) \in \mathcal{L} - \mathcal{P}$.  

**Proof.** The following proof is based on an argument by contradiction. Thus we will suppose that there is no integer $m$ such that $D^m\varphi(x) \in \mathcal{L} - \mathcal{P}$. To show that this assumption is untenable, we will establish the desired contradiction by proving that $\rho(\varphi) \geq 2$.

Since differentiation does not increase the number of nonreal zeros of a function $\varphi(x)$ in $\mathcal{L} - \mathcal{P}^*$ (cf. Section 2), we may assume, without loss of
generality, that \( Z_c(\varphi^{(k)}(x)) = 2d, \ d > 0, \) for \( k \geq 0. \) Let \( \alpha(j, k) + i\beta(j, k), \ \beta(j, k) > 0, \ 1 \leq j \leq d, \) denote the nonreal zeros of \( \varphi^{(k)}(x). \) (Here and in the sequel it will be convenient for us to employ the notation \( \alpha(j, k) = a_{j,k} \) for terms of sequences indexed by double subscripts.) Now by the Jensen-Nagy-Walsh theorem we know that for any positive integer \( k, \) the nonreal zeros of \( D^{2^k + 1} \varphi(x) \) lie in the union of the \( 2^k \)-th Jensen ellipses determined by the nonreal zeros of \( D^{2^k} \varphi(x). \) Therefore, for any positive integer \( k \) and for any \( j, 1 \leq j \leq d, \) there is an integer \( j', 1 \leq j' \leq d, \) such that

\[
(3.44) \quad |\alpha(j, 2^{k+1}) - \alpha(j', 2^k)| \leq \beta(j', 2^k)2^{k/2}.
\]

But by the proposition, \( \beta(j, k) \leq c_1 A_1/k, \ k \geq 1, \ 1 \leq j \leq d, \) where \( c_1 \) is a constant independent of \( k. \) (Henceforth, to avoid repetition, \( c_1, \ldots, c_6 \) will denote constants which are independent of \( k. \)) Hence, inequality (3.44) becomes

\[
(3.45) \quad |\alpha(j, 2^{k+1}) - \alpha(j', 2^k)| \leq c_1 A_2 2^{-k/2}, \quad k \geq 0.
\]

We next show that the sequence \( \{ A_2^k \}_{k=0}^{\infty} \) is bounded. To this end we recall the definition of \( A_k \) (see (3.16)), and consider the inequalities

\[
|\alpha(j, 2^{k+1}) + a| \leq |\alpha(j, 2^{k+1}) - \alpha(j', 2^k)| + |\alpha(j', 2^k) + a|
\leq c_1 A_2^k 2^{-k/2} + A_2^k
= A_2^k(c_1 2^{-k/2} + 1).
\]

Hence, it follows that

\[
A_{2^{k+1}} \leq A_2^k(1 + c_1 2^{-k/2})
\]

and

\[
(3.46) \quad A_2^k \leq A_1 \prod_{i=0}^{k-1} (1 + c_1 2^{-i/2}) \leq c_2.
\]

Thus, in particular, we obtain for \( 1 \leq j \leq d, \)

\[
(3.47) \quad \beta(j, 2^k) \leq c_1 A_2^k 2^{-k} \leq c_3 2^{-k}, \quad k \geq 0.
\]

Moreover, by (3.46) inequality (3.45) becomes

\[
|\alpha(j, 2^{k+1}) - \alpha(j', 2^k)| \leq c_4 2^{-k/2}.
\]

Next let

\[
r_k = \sum_{i=k}^{\infty} c_4 2^{-i/2} = c_5 2^{-k/2},
\]

and set

\[
G_k = \bigcup_{j=1}^{d} I(j, k),
\]
where \( I(j, k) \) denotes the closed interval
\[
I(j, k) = \left[ \alpha(j, 2^k) - r_k, \alpha(j, 2^k) + r_k \right], \quad 1 \leq j \leq d, \quad k \geq 0.
\]
Since \( G_{k+1} \subseteq G_k \) and \( r_k \to 0 \) as \( k \to \infty \),
\[
G = \bigcap_{k=0}^{\infty} G_k \neq \emptyset,
\]
and \( G \) consists of at most \( d \) real numbers: \( G = \{y_1, \ldots, y_p\} \), \( p \leq d \). Since \( y_1 \) is in \( G \), for each nonnegative integer \( k \) there is an integer \( j, 1 \leq j \leq d \), such that \( y_1 \) belongs to \( I(j, k) \). Thus for each \( k \geq 0 \) there is a \( j, 1 \leq j \leq d \), such that
\[
|\alpha(j, 2^k) - y_1| \leq r_k = c_3 2^{-k/2}.
\]
Next, if \( n \) is an integer such that \( 2^{k-1} \leq n \leq 2^k \), then by (3.47) and by the Jensen-Nagy-Walsh theorem we obtain the estimate
\[
\beta(j, n) \leq c_3 2^{-k+1} = c_6 2^{-k} \leq c_6 n^{-1}, \quad 1 \leq j \leq d.
\]
Furthermore, if \( 2^{k-1} \leq n \leq 2^k \), then by the Jensen-Nagy-Walsh theorem and by (3.49) we have that there is an integer \( j', 1 \leq j' \leq d \) such that
\[
|\alpha(j, 2^k) - \alpha(j', n)| \leq \beta(j', n)(2^k - n)^{1/2} \leq c_6 n^{-1/2}.
\]
Therefore, it follows from (3.48) and (3.50) that for each positive integer \( n \), there is an integer \( j, 1 \leq j \leq d \), such that
\[
|\alpha(j, n) - y_1| \leq c_7 n^{-1/2},
\]
where \( c_7 \) is a constant independent of \( n \). Hence, if we combine (3.49) and (3.51) we obtain that for each positive integer \( n \), there is an integer \( j, 1 \leq j \leq d \), such that
\[
|\alpha(j, n) \pm i\beta(j, n) - y_1| \leq c_8 n^{-1/2},
\]
where \( c_8 \) is a constant independent of \( n \).

Finally, if \( \rho = \rho(\varphi) < 2 \), then we can select \( \lambda \) such that \( \rho < \lambda < 2 \). But then by Ålander’s theorem there are infinitely many \( n \) such that
\[
|\alpha(j, n) + i\beta(j, n) - y_1| > (\log 2)n^{-1 + 1/\lambda}, \quad 1 \leq j \leq d.
\]
Since \( 1 - 1/\lambda < 1/2 \), (3.53) contradicts (3.52), and thus the proof of the theorem is complete.

4. Some remarks and open problems

In conclusion, we will comment here on the foregoing results and we will cite some open questions. To begin with, we note that Theorem 1 is a result
concerning the zeros of the logarithmic derivative of a function in $\mathcal{L} - \mathcal{P}^*$. Now it is well-known that the behavior of the logarithmic derivative of an entire function $f$ plays an essential role in the study of the distribution of its zeros. Therefore, in light of Theorem 1 it is natural to consider here the following related question of Gauss [N2]. Let $p(x)$ be a real polynomial of degree $n$, $n \geq 2$, and suppose that $p(x)$ has exactly $2d$ nonreal zeros, $0 \leq 2d \leq n$. Then Gauss' problem is to find a relationship between the number $2d$ and the number of real zeros of the rational function

\begin{equation}
Q(x) = D\left(\frac{p'}{p}(x)\right), \quad D = \frac{d}{dx}.
\end{equation}

If $p(x)$ has only real zeros, then $Q(x) < 0$ and consequently in this special case the answer is clear. We also know by Theorem 1 that if for some $\gamma \in \mathbb{R}$, the polynomial $\gamma p(x) + p'(x)$ has only real zeros then $Q(x)$ has precisely $2d$ real zeros. Gauss' problem has been studied by several authors (see, for example, Nagy [N2]) and it appears that this problem has remained open since 1836. On the basis of our preliminary investigations we conjecture that

\begin{equation}
Z_R(Q(x)) \leq 2d,
\end{equation}

where $Q(x)$ is given by (4.1).

We next consider Theorem 3. In view of the work of Fourier and Pólya [P2], the result of Theorem 3 can also be expressed in terms of the Fourier critical points (see the definition below) of a function $\varphi$ in $\mathcal{L} - \mathcal{P}^*$, if $\rho(\varphi) < 2$. A real number $t$ is said to be a $k$-fold Fourier critical point of order $p + 1$ of $\varphi$, $\varphi \in \mathcal{L} - \mathcal{P}^*$, if

\begin{align*}
\varphi^{(p)}(t) &\neq 0, \\
\varphi^{(p+1)}(t) = \varphi^{(p+2)}(t) = \cdots = \varphi^{(p+m)}(t) = 0, \\
\varphi^{(p+1+m)}(t) &\neq 0, \quad p \geq 0, \quad m \geq 1,
\end{align*}

where the multiplicity $k$ is determined by

\begin{enumerate}
  \item $k = \frac{m}{2}$, \quad if $m$ is even,
  \item $k = \frac{m + 1}{2}$, \quad if $m$ is odd and $\varphi^{(p)}(t)\varphi^{(p+m+1)}(t) > 0$,
  \item $k = \frac{m - 1}{2}$, \quad if $m$ is odd and $\varphi^{(p)}(t)\varphi^{(p+m+1)}(t) < 0$.
\end{enumerate}

The totality of all Fourier critical points of $\varphi$, of all orders, counting multiplicities, is called the number of Fourier critical points of $\varphi$. We note, in particular,
that if a function \( \varphi \) in \( L - P^* \) has a positive local minimum or a negative local maximum at a point \( t_0 \), then \( t_0 \) is a Fourier critical point of \( \varphi \). Therefore, by Pólya’s result [P2, Theorem I, pp. 25–26], Theorem 3 can be cast in the following equivalent form.

**Theorem 4.** Let \( \varphi \in L - P^* \) and suppose that \( \rho(\varphi) < 2 \). Then the number of pairs of nonreal zeros of \( \varphi \) is equal to the number of Fourier critical points of \( \varphi \).

In reference to the proof of Theorem 3, we emphasize that our argument does not yield an estimate of the integer \( m \) for which \( D^m \varphi(x) \) has only real zeros. Finally, we remark that the more general conjecture (see Pólya [P4, footnote, p. 182]) involving an arbitrary function of order 2 in \( L - P^* \) remains open.*

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**References**


*Added in proof. This is partially solved in "Zeros of derivatives of entire functions" by the same authors, scheduled to appear in Proc. A.M.S.*


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