

KRONECKER COEFFICIENTS FOR SOME NEAR-RECTANGULAR PARTITIONS

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ABSTRACT. We give formulae for computing Kronecker coefficients occurring in the expansion of $s_\mu * s_\nu$, where both μ and ν are nearly rectangular, and have smallest parts equal to either 1 or 2. In particular, we study $s_{(n,n-1,1)} * s_{(n,n)}$, $s_{(n-1,n-1,1)} * s_{(n,n-1)}$, $s_{(n-1,n-1,2)} * s_{(n,n)}$, $s_{(n-1,n-1,1,1)} * s_{(n,n)}$ and $s_{(n,n,1)} * s_{(n,n,1)}$. Our approach relies on the interplay between manipulation of symmetric functions and the representation theory of the symmetric group, mainly employing the Pieri rule and a useful identity of Littlewood. As a consequence of these formulae, we also derive an expression enumerating certain standard Young tableaux of bounded height, in terms of the Motzkin and Catalan numbers.

An outstanding open problem in algebraic combinatorics is to derive a combinatorial formula to compute the Kronecker product of two Schur functions. Given partitions λ, μ and ν , the Kronecker coefficients, $g_{\mu\nu}^\lambda$, occur in the decomposition of the Kronecker product $s_\mu * s_\nu$ of Schur functions in the Schur basis.

$$s_\mu * s_\nu = \sum_{\lambda} g_{\mu\nu}^\lambda s_\lambda$$

Alternatively, these coefficients can also be defined as the multiplicities of the irreducible representations of the symmetric group in the tensor product of two irreducible representations of the symmetric group. This interpretation immediately implies that the Kronecker coefficients are non-negative integers suggesting that there should be a combinatorial rule to compute these coefficients. However, to date, there is no satisfactory positive combinatorial formula for the Kronecker product of two Schur functions.

Besides the intrinsic interest in the problem, the motivation for discovering a combinatorial formula is the impact beyond algebraic combinatorics. For example, the Kronecker coefficients arise in quantum information theory and quantum computation [13, 14, 34]. The problem of computing them combinatorially has received major impetus since they are of prime importance in Geometric Complexity Theory, a program of Mulmuley aimed at resolving the P vs NP problem [24]. In other applications, these coefficients have been used to show the strict unimodality of q -binomial numbers by Pak and Panova [25].

Attempts have been made to understand different aspects of these coefficients, for example, special cases [5, 6, 10, 27, 28, 33], asymptotics [1, 2], stability [9, 32], the complexity of

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computing them and conditions which guarantee that they are non-zero [12]. Recently, a combinatorial rule was given by Blasiak [7] for computing $s_\mu * s_\nu$ where at least one of μ and ν is a hook shape. Finally, a certain variant, called the reduced Kronecker coefficients, has also been studied in [8, 9].

The aim of this article is to derive explicit combinatorial formulae for Kronecker coefficients corresponding to partitions of *near-rectangular shape*, i.e., partitions such that nearly all their parts are equal. Kronecker coefficients indexed by such partitions are conducive to manipulation, as demonstrated in [10, 11, 13, 23, 34]. The organization of this article is as follows. In Section 1, we equip the reader with the required background on symmetric functions and a brief overview of relevant results. In Sections 2, 3, 4, 5 and 6, we prove combinatorial formulae for the Kronecker coefficients appearing in the products $s_{(n,n-1,1)} * s_{(n,n)}$, $s_{(n-1,n-1,1)} * s_{(n,n-1)}$, $s_{(n-1,n-1,2)} * s_{(n,n)}$, $s_{(n-1,n-1,1,1)} * s_{(n,n)}$ and $s_{(n,n,1)} * s_{(n,n,1)}$ respectively. The interested reader can find more detailed proofs in [31]. Using the results obtained, we give a closed formula for the number of standard Young tableaux of height exactly 5 and smallest part equal to 1. This is stated in Theorem 7.4 in Section 7.

1. BACKGROUND

We will start by defining some of the combinatorial structures that we will be encountering. All the central notions introduced in this section are covered in more detail in [22, 29, 30]. Our first definition concerns the notion of partition.

1.1. Partitions. A *partition* λ is a finite list of positive integers $(\lambda_1, \dots, \lambda_k)$ satisfying $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$. The integers appearing in the list are called the *parts* of the partition. Given a partition $\lambda = (\lambda_1, \dots, \lambda_k)$, the *size* $|\lambda|$ is defined to be $\sum_{i=1}^k \lambda_i$. The number of parts of λ is called the *length*, and is denoted by $l(\lambda)$. If λ is a partition satisfying $|\lambda| = n$, then we denote this by $\lambda \vdash n$. Conventionally, there is a unique partition of size and length 0, and we denote it by \emptyset .

We will be depicting a partition using its *Ferrers diagram* (or *Young diagram*). Given a partition $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$, the Ferrers diagram of λ , also denoted by λ , is the left-justified array of n boxes, with λ_i boxes in the i -th row. We will be using the English convention, i.e. the rows are numbered from top to bottom and the columns from left to right. We refer to the box in the i -th row and j -th column by the ordered pair (i, j) . Finally, the *transpose*, λ^t , of a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ is the partition obtained by transposing the Ferrers diagram of λ . Thus, for example, the transpose of the partition $\lambda = (5, 3, 3, 1)$ is $\lambda^t = (4, 3, 3, 1, 1)$. The *hooklength* associated to the box (i, j) , denoted by $h_{(i,j)}$ is the number $\lambda_i - j + \lambda_j^t - i + 1$.

If λ and μ are partitions such that $\mu \subseteq \lambda$, i.e., $l(\mu) \leq l(\lambda)$ and $\mu_i \leq \lambda_i$ for all $i = 1, 2, \dots, l(\mu)$, then the *skew shape* λ/μ is obtained by removing the first μ_i boxes from the i -th row of the Ferrers diagram of λ for $1 \leq i \leq l(\mu)$. The *size* of the skew shape λ/μ , denoted by $|\lambda/\mu|$, is equal to the number of boxes in the skew shape, i.e., $|\lambda| - |\mu|$.

Let μ and λ be partitions. We say that $\mu \prec \lambda$ if μ can be obtained by subtracting 1 from some part of λ .

Example 1.1. Shown below is the Ferrers diagram of $\lambda = (5, 3, 3, 1)$ (left) and the respective hooklengths associated with each box (right).

8	6	5	2	1
5	3	2		
4	2	1		
1				

Now, we will define some statistics on partitions that we will need to state our results, especially in Sections 4, 5 and 6. We will denote the number of distinct parts in a partition λ by d_λ , while $d_{\lambda,2}$ will denote the number of parts of λ from which 2 can be subtracted so that whatever remains (once the tail of zeroes has been removed) is still partition. Finally, we denote by R_λ the number of distinct parts of λ that occur at least twice. For example, consider $\lambda = (6, 5, 3, 3, 3, 2, 2)$. Clearly, $d_\lambda = 4$ and $R_\lambda = 2$. Observe that on subtracting 2 from the part equal to 5 in λ , we obtain $(6, 3, 3, 3, 3, 2, 2)$ which is a partition while on subtracting 2 from the rightmost part equal to 2 in λ , we get $(6, 5, 3, 3, 3, 2, 0)$ which becomes a partition once we remove the 0 in the tail. Hence $d_{\lambda,2} = 2$.

Given a partition λ , define a new partition λ' as follows.

$$\lambda' = \begin{cases} \lambda & l(\lambda) \leq 4 \\ (\lambda_1, \dots, \lambda_4) & l(\lambda) > 4. \end{cases}$$

Thus, for example, if $\lambda = (4, 3, 1)$ then so is λ' , but if $\lambda = (5, 5, 3, 3, 2, 1)$ then $\lambda' = (5, 5, 3, 3)$. Now given a partition λ , define O_λ and E_λ to be the number of odd and even parts in λ' respectively. Also, define O'_λ and E'_λ to be the number of distinct odd parts and distinct even parts in λ' respectively.

To illustrate these definitions, we give an example. Consider $\lambda = (4, 4, 3, 2, 1)$. Then the number of even parts in $\lambda' = (4, 4, 3, 2)$ is 3. Hence $E_\lambda = 3$, but notice the the number of distinct even parts in λ' is just 2, i.e. $E'_\lambda = 2$. Note also that $O_\lambda = O'_\lambda = 1$.

1.2. Semistandard Young tableaux. Given partitions λ and μ such that $\mu \subseteq \lambda$, a *semistandard Young tableau* (SSYT) of *shape* λ/μ is a filling of the boxes of the skew shape λ/μ with positive integers satisfying the condition that entries increase weakly along each row from left to right and increase strictly along each column from top to bottom.

A *standard Young tableau* (SYT) of shape λ/μ is an SSYT in which the entries in the filling are distinct elements of $\{1, 2, \dots, |\lambda/\mu|\}$. We denote by $SSYT(\lambda/\mu)$ the set of all SSYTs of shape λ/μ . As a matter of convention, an SSYT of shape λ/\emptyset will be referred to as an SSYT of shape λ . The *height* of an SSYT of shape λ is defined to be $l(\lambda)$.

The number of SYTs of shape $\lambda \vdash n$ will be denoted by f_λ , and it can be easily calculated by the following *hooklength formula* of Frame, Robinson and Thrall.

Theorem 1.2. [17, Theorem 1] *Given a partition λ of n ,*

$$f_\lambda = \frac{n!}{\prod_{(i,j) \in \lambda} h(i,j)}.$$

Finally, given a skew shape λ/μ , we will associate a monomial x^T to every $T \in SSYT(\lambda/\mu)$ in the following manner

$$x^T = \prod_{(i,j) \in \lambda/\mu} x_{T(i,j)},$$

where $T_{(i,j)}$ denotes the entry in the i -th row and j -th column of T .

Example 1.3. *An SSYT (left) and an SYT (right) of shape $\lambda = (4, 3, 1, 1)$ are shown below.*

1	3	3	3
2	4	4	
5			
6			

1	3	5	7
2	4	8	
6			
9			

The monomial associated with the SSYT on the left is $x_1x_2x_3^3x_4^2x_5x_6$.

1.3. Symmetric functions. We will denote the algebra of symmetric functions by Λ . It is the algebra freely generated over \mathbb{Q} by countably many commuting variables $\{p_1, p_2, \dots\}$. Assigning the degree i to p_i (and then extending this multiplicatively) gives Λ the structure of a graded algebra. A basis for the degree n component of Λ , denoted by Λ^n , is given by the *power sum symmetric functions* of degree n ,

$$\{p_\lambda = p_{\lambda_1} \cdots p_{\lambda_k} : \lambda = (\lambda_1, \dots, \lambda_k) \vdash n\}.$$

A concrete realization of Λ is obtained by embedding $\Lambda = \mathbb{Q}[p_1, p_2, \dots]$ in $\mathbb{Q}[[x_1, x_2, \dots]]$, i.e., the ring of formal power series in countably many commuting indeterminates $\{x_1, x_2, \dots\}$, under the identification (extended multiplicatively)

$$p_i \longmapsto \sum_{j \geq 1} x_j^i.$$

Thus we can consider symmetric functions as formal power series f in the $\{x_1, x_2, \dots\}$ such that $f(x_{\pi(1)}, x_{\pi(2)}, \dots) = f(x_1, x_2, \dots)$ for every permutation π of the positive integers \mathbb{N} . It is with this viewpoint that we will define a very important class of symmetric functions next.

1.4. Schur functions. We start by defining the skew Schur functions combinatorially.

Definition 1.4. *Given a skew shape λ/μ , the skew Schur function of shape λ/μ , $s_{\lambda/\mu}$, is the formal power series*

$$s_{\lambda/\mu} = \sum_{T \in SSYT(\lambda/\mu)} x^T.$$

If $\mu = \emptyset$, then $\lambda/\mu = \lambda$, and we call s_λ the Schur function of shape λ .

Though not evident from this definition, skew Schur functions are actually symmetric functions and the elements of the set $\{s_\lambda : \lambda \vdash n\}$ form a basis for Λ^n . We can further equip this space with an inner product $\langle \cdot, \cdot \rangle_{\Lambda^n}$, called the *Hall inner product*. It is defined by setting $\langle s_\lambda, s_\mu \rangle_{\Lambda^n} = \delta_{\lambda\mu}$, where $\delta_{\lambda\mu} = 1$ if $\lambda = \mu$ and 0 otherwise, and then defining the inner product for any $f, g \in \Lambda^n$ by linear extension. One can extend this to an inner product on Λ , in which case we will refer to it as $\langle \cdot, \cdot \rangle_\Lambda$. It satisfies the following property with respect to skew Schur functions.

$$\langle s_\mu s_\nu, s_\lambda \rangle_\Lambda = \langle s_\nu, s_{\lambda/\mu} \rangle_\Lambda$$

One place where this inner product arises is when we wish to multiply together two Schur functions and express the result in terms of the basis of Schur functions. The coefficients so obtained are called the *Littlewood-Richardson coefficients* and are given by the *Littlewood-Richardson rule* which takes the form of an algorithm that counts semistandard Young tableaux satisfying certain properties. More precisely, given partitions μ and ν , we have an expansion as follows

$$s_\mu s_\nu = \sum_{\lambda} c_{\mu,\nu}^\lambda s_\lambda$$

where the sum is over all λ such that μ is contained in λ . Here the $c_{\mu,\nu}^\lambda$ are the Littlewood-Richardson coefficients and in terms of the inner product on Λ

$$c_{\mu,\nu}^\lambda = \langle s_\mu s_\nu, s_\lambda \rangle_\Lambda = \langle s_\nu, s_{\lambda/\mu} \rangle_\Lambda.$$

We will only require special cases of the Littlewood-Richardson rule, which describe the multiplication of a Schur function with a Schur function of one row or one column. These cases are collectively called the *Pieri rule* but before we state the rule we need to describe certain skew shapes. A skew shape λ/μ is called a *horizontal strip* if it does not contain boxes in the same column, and is called a *vertical strip* if it does not contain boxes in the same row.

Theorem 1.5 (Pieri rule). *If μ is a partition, then*

$$\begin{aligned} s_\mu s_{(n)} &= \sum_{\substack{\nu \vdash |\mu|+n \\ \nu/\mu = \text{horizontal strip of size } n}} s_\nu \\ s_\mu s_{(1^n)} &= \sum_{\substack{\nu \vdash |\mu|+n \\ \nu/\mu = \text{vertical strip of size } n}} s_\nu. \end{aligned}$$

1.5. The Kronecker product of Schur functions. In this section we will outline how the Kronecker coefficients arise in the representation theory of the symmetric group. Given $\mu \vdash n$, let V^μ denote the irreducible representation of \mathfrak{S}_n indexed by μ , whose dimension equals f_μ , and let the corresponding character be denoted by χ_μ . Then the pointwise product $\chi_\mu \chi_\nu$ is the character of the \mathfrak{S}_n -representation $V^\mu \otimes V^\nu$. Let $g_{\mu\nu}^\lambda$ denote the multiplicity of V^λ in $V^\mu \otimes V^\nu$. That is, $g_{\mu\nu}^\lambda = \langle \chi_\mu \chi_\nu, \chi_\lambda \rangle_{CF^n}$ where $\langle \cdot, \cdot \rangle_{CF^n}$ denotes the standard inner

product on the space of class functions CF^n of \mathfrak{S}_n . We shall now provide a direct definition for the Kronecker coefficients using the Kronecker product of two Schur functions (these two definitions may be seen to coincide as the Schur function, s_λ , is the cycle-indicator generating function for the irreducible character χ_λ of $\mathfrak{S}_{|\lambda|}$).

The *cycle type* of a permutation σ is the partition obtained by ordering the cycle lengths occurring in the cycle decomposition of σ in weakly decreasing order. Given a partition λ , let z_λ denote the number of permutations in $\mathfrak{S}_{|\lambda|}$ commuting with a fixed permutation of cycle type λ . This given, the *Kronecker product*, $*$, on Λ is defined implicitly by defining it on the basis of power sum symmetric functions by

$$\frac{p_\lambda}{z_\lambda} * \frac{p_\mu}{z_\mu} = \delta_{\lambda\mu} \frac{p_\lambda}{z_\lambda},$$

and then extending it linearly. With this definition, it transpires that the *Kronecker coefficients* $g_{\mu\nu}^\lambda$ are given by

$$g_{\mu\nu}^\lambda = \langle \chi_\mu \chi_\nu, \chi_\lambda \rangle_{CF^n} = \langle s_\mu * s_\nu, s_\lambda \rangle_{\Lambda^n} \quad (1)$$

where λ , μ and ν are partitions of the same size n . The Kronecker product also satisfies the useful identities

$$s_\mu * s_\nu = s_\nu * s_\mu \text{ and } s_\mu * s_\nu = s_{\nu^t} * s_{\mu^t}.$$

Moreover, if $\mu, \nu \vdash n$ then $g_{\mu\nu}^{(n)} = g_{\mu\nu^t}^{(1^n)} = \delta_{\mu\nu}$.

Remark. *From now on, we shall only consider the Hall inner product of symmetric functions and thus \langle, \rangle is to be interpreted as \langle, \rangle_Λ .*

Before we recall the relevant results on Kronecker products we will need later, we fix some notation. Given a positive integer n , let

$$P_n = \{\lambda \vdash 2n : l(\lambda) \leq 4 \text{ and } \lambda \text{ has either all parts even or } l(\lambda) = 4 \text{ and all parts odd}\},$$

$$Q_n = \{\lambda \vdash 2n : l(\lambda) \leq 4 \text{ and exactly two parts of } \lambda \text{ are odd}\}.$$

This given, let

$$P = \bigcup_{n \geq 0} P_n \text{ and } Q = \bigcup_{n \geq 0} Q_n,$$

and it is clear that $P \cup Q$ is the set of all partitions of even size and length at most 4.

We will use the *Knuth bracket* for giving truth values to statements.

$$((S)) = \begin{cases} 1 & S \text{ is a true statement} \\ 0 & \text{otherwise.} \end{cases}$$

Now we are in a position to state the results of interest to us. The computation of $s_{(n,n)} * s_{(n,n)}$ is one such result. This computation originally arose while solving a mathematical physics problem related to resolving the interference of 4 qubits [34]. It appeared first in [18] in the form as shown below. It was proven again in [10]. The result states the following.

Theorem 1.6. [18, Theorem I.6] *Given a positive integer n ,*

$$s_{(n,n)} * s_{(n,n)} = \sum_{\lambda \in P_n} s_\lambda.$$

This characterization is different from earlier characterizations as it explicitly states which partitions have non-zero coefficients and further establishes that the coefficients are all either 0 or 1 without giving a combinatorial rule.

Using the result of [18] as inspiration, a characterization of the Kronecker product of $s_{(n,n)} * s_{(n+k,n-k)}$ for $k \geq 0$ was obtained in [10]. Since we do not need the full strength of that result, we will only state the $k = 1$ case.

Theorem 1.7. [10, Corollary 3.6] *Given a positive integer n ,*

$$s_{(n+1,n-1)} * s_{(n,n)} = \sum_{\lambda \in Q_n} s_\lambda.$$

A result of Littlewood that we will frequently use, and simplifies many of our calculations, is the following.

Theorem 1.8. [21] *Let α, β and γ be partitions such that $|\alpha| + |\beta| = |\gamma|$. Then,*

$$(s_\alpha s_\beta) * s_\gamma = \sum_{\delta \vdash |\beta|} \sum_{\eta \vdash |\alpha|} c_{\eta, \delta}^\gamma (s_\eta * s_\alpha) (s_\delta * s_\beta)$$

where the $c_{\eta, \delta}^\gamma$ are Littlewood-Richardson coefficients.

Using this identity of Littlewood in conjunction with Theorem 1.6, one can prove the following corollary, present in the following form in [10].

Corollary 1.9. [10, Corollary 4.1] *Given a positive integer n ,*

$$s_{(n,n-1)} * s_{(n,n-1)} = \sum_{\substack{\lambda \vdash 2n-1 \\ l(\lambda) \leq 4}} s_\lambda.$$

We will need one final result which, given partitions μ and ν , helps in identifying certain partitions λ for which $g_{\mu\nu}^\lambda = 0$. Below, $\mu \cap \nu$ denotes the partition obtained by intersecting the corresponding Ferrers diagrams once their top left corners are aligned. Clausen and Meier [15] and Dvir [16] proved the following theorem.

Theorem 1.10. *Let μ, ν be partitions of n . Then*

$$\begin{aligned} \max \{ \lambda_1 : g_{\mu\nu}^\lambda \neq 0 \text{ for some } \lambda = (\lambda_1, \dots, \lambda_{l(\lambda)}) \} &= |\mu \cap \nu|, \\ \max \{ l(\lambda) : g_{\mu\nu}^\lambda \neq 0 \text{ for some } \lambda = (\lambda_1, \dots, \lambda_{l(\lambda)}) \} &= |\mu \cap \nu^t|. \end{aligned}$$

The import of this theorem can be gauged by the fact that it already implies that if μ and ν are partitions each with at most two rows, then $g_{\mu\nu}^\lambda = 0$ for all λ such that $l(\lambda) \geq 5$.

2. THE KRONECKER COEFFICIENT $g_{(n,n-1,1)(n,n)}^\theta$ FOR $n \geq 2$

We will now derive an explicit characterization of the coefficients arising in the Kronecker product of $s_{(n,n-1,1)}$ and $s_{(n,n)}$. Observe that the Pieri rule (Theorem 1.5) implies that

$$s_{(n,n-1,1)} = s_{(n,n-1)}s_{(1)} - s_{(n,n)} - s_{(n+1,n-1)}.$$

Since we are interested in computing $g_{(n,n-1,1)(n,n)}^\theta$ where $\theta \vdash 2n$, we will compute $\langle s_{(n,n-1,1)} * s_{(n,n)}, s_\theta \rangle$ by (1). Using the equation above, we obtain

$$\begin{aligned} \langle s_{(n,n-1,1)} * s_{(n,n)}, s_\theta \rangle &= \langle (s_{(n,n-1)}s_{(1)}) * s_{(n,n)}, s_\theta \rangle \\ &\quad - \langle (s_{(n,n)} + s_{(n+1,n-1)}) * s_{(n,n)}, s_\theta \rangle. \end{aligned} \quad (2)$$

We will evaluate the inner products appearing on the right hand side of (2) individually. Theorem 1.8 implies that

$$\begin{aligned} (s_{(n,n-1)}s_{(1)}) * s_{(n,n)} &= \sum_{\delta \vdash 1} \sum_{\eta \vdash 2n-1} c_{\eta,\delta}^{(n,n)}(s_\eta * s_{(n,n-1)})(s_\delta * s_{(1)}) \\ &= \sum_{\eta \vdash 2n-1} c_{\eta,(1)}^{(n,n)}(s_\eta * s_{(n,n-1)})(s_{(1)} * s_{(1)}). \end{aligned}$$

The Pieri rule yields that $c_{\eta,(1)}^{(n,n)} \neq 0$ if and only if $\eta = (n, n-1)$, in which case $c_{(n,n-1),(1)}^{(n,n)} = 1$. Since $s_{(1)} * s_{(1)} = s_{(1)}$, we conclude that

$$(s_{(n,n-1)}s_{(1)}) * s_{(n,n)} = s_{(1)}(s_{(n,n-1)} * s_{(n,n-1)}).$$

This reduces (2) to

$$\begin{aligned} \langle s_{(n,n-1,1)} * s_{(n,n)}, s_\theta \rangle &= \langle s_{(1)}(s_{(n,n-1)} * s_{(n,n-1)}), s_\theta \rangle - \langle (s_{(n,n)} + s_{(n+1,n-1)}) * s_{(n,n)}, s_\theta \rangle \\ &= \langle s_{(n,n-1)} * s_{(n,n-1)}, s_{\theta/(1)} \rangle - \langle (s_{(n,n)} + s_{(n+1,n-1)}) * s_{(n,n)}, s_\theta \rangle \\ &= \left\langle \sum_{\substack{\lambda \vdash 2n-1 \\ l(\lambda) \leq 4}} s_\lambda, s_{\theta/(1)} \right\rangle - \left\langle \sum_{\substack{\lambda \vdash 2n \\ l(\lambda) \leq 4}} s_\lambda, s_\theta \right\rangle, \end{aligned} \quad (3)$$

where in arriving at the last step in the above sequence, we have made use of Corollary 1.9, Theorem 1.6 and Theorem 1.7. Notice $\left\langle \sum_{\lambda \vdash 2n, l(\lambda) \leq 4} s_\lambda, s_\theta \right\rangle$ is 1 if $l(\theta) \leq 4$ and 0 otherwise. So

we will focus on evaluating $\left\langle \sum_{\lambda \vdash 2n-1, l(\lambda) \leq 4} s_\lambda, s_{\theta/(1)} \right\rangle$. The Pieri rule implies that

$$s_{\theta/(1)} = \sum_{\theta^- \prec \theta} s_{\theta^-}.$$

Note the crucial fact that the number of terms appearing on the right hand side of the equation above is equal to the number of distinct parts in the partition θ , i.e., d_θ .

If $\langle s_{(n,n-1,1)} * s_{(n,n)}, s_\theta \rangle \neq 0$, then by Theorem 1.10 we have that $l(\theta) \leq 5$, as $|(n, n-1, 1) \cap (n, n)^t| \leq 5$. We will complete the computation using case analysis dependent on the length of θ .

2.1. Case I: $l(\theta) = 5$. If $l(\theta) = 5$ and $\theta_5 \geq 2$, then $s_{\theta/(1)}$ is sum of terms of the form s_δ with $l(\delta) = 5$. The right hand side of (3) clearly implies that the coefficient of s_θ in $s_{(n,n-1,1)} * s_{(n,n)}$ is 0 in this instance. If $\theta_5 = 1$, then $s_{\theta/(1)} = s_{\theta'} + \text{sum of terms of the form } s_\delta \text{ where } l(\delta) = 5$.

This in turn means that $\langle \sum_{\substack{\lambda \vdash 2n-1 \\ l(\lambda) \leq 4}} s_\lambda, s_{\theta/(1)} \rangle = 1$. Thus, if $l(\theta) = 5$,

$$\langle s_{(n,n-1,1)} * s_{(n,n)}, s_\theta \rangle = \begin{cases} 1 & \theta_5 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

2.2. Case II: $l(\theta) \leq 4$. We know that if $l(\theta) \leq 4$, then $\langle \sum_{\lambda \vdash 2n, l(\lambda) \leq 4} s_\lambda, s_\theta \rangle = 1$. The following computation helps us complete this case.

$$\begin{aligned} \langle \sum_{\substack{\lambda \vdash 2n-1 \\ l(\lambda) \leq 4}} s_\lambda, s_{\theta/(1)} \rangle &= \langle \sum_{\substack{\lambda \vdash 2n-1 \\ l(\lambda) \leq 4}} s_\lambda, \sum_{\theta^- \prec \theta} s_{\theta^-} \rangle \\ &= d_\theta. \end{aligned}$$

Thus, using (3), we get that for $l(\theta) \leq 4$,

$$\langle s_{(n,n-1,1)} * s_{(n,n)}, s_\theta \rangle = d_\theta - 1.$$

On collecting the results of the two cases together, we obtain the following theorem.

Theorem 2.1. *Let $\lambda = (n, n-1, 1)$, $\mu = (n, n)$ and $\theta \vdash 2n$. Then the Kronecker coefficients labelled by these partitions are as follows.*

$$g_{\lambda\mu}^\theta = \begin{cases} 1 & l(\theta) = 5, \theta_5 = 1 \\ d_\theta - 1 & l(\theta) \leq 4 \\ 0 & \text{otherwise.} \end{cases}$$

Example 2.2. *Theorem 2.1 gives the following expansion for $s_{(4,3,1)} * s_{(4,4)}$.*

$$\begin{aligned} s_{(4,3,1)} * s_{(4,4)} &= s_{(2,2,2,1,1)} + s_{(3,2,1,1,1)} + 2s_{(3,2,2,1)} + s_{(3,3,1,1)} + s_{(3,3,2)} \\ &\quad + s_{(4,1,1,1,1)} + 2s_{(4,2,1,1)} + s_{(4,2,2)} + 2s_{(4,3,1)} + s_{(5,1,1,1)} \\ &\quad + 2s_{(5,2,1)} + s_{(5,3)} + s_{(6,1,1)} + s_{(6,2)} + s_{(7,1)}. \end{aligned}$$

3. THE KRONECKER COEFFICIENT $g_{(n-1,n-1,1)(n,n-1)}^\theta$ FOR $n \geq 2$

Using techniques similar to the previous case, we can explicitly compute the coefficients arising in the Kronecker product of $s_{(n-1,n-1,1)}$ and $s_{(n,n-1)}$. Again, the Pieri rule implies that

$$s_{(n-1,n-1,1)} = s_{(1)}s_{(n-1,n-1)} - s_{(n,n-1)}. \quad (4)$$

An application of Theorem 1.8 gives

$$\begin{aligned} (s_{(1)}s_{(n-1,n-1)}) * s_{(n,n-1)} &= \sum_{\delta \vdash 1} \sum_{\eta \vdash 2n-2} c_{\eta,\delta}^{(n,n-1)} (s_{\eta} * s_{(n-1,n-1)})(s_{\delta} * s_{(1)}) \\ &= \sum_{\eta \vdash 2n-2} c_{\eta,(1)}^{(n,n-1)} (s_{\eta} * s_{(n-1,n-1)})(s_{(1)} * s_{(1)}). \end{aligned}$$

The Pieri rule dictates that the only cases where $c_{\eta,(1)}^{(n,n-1)} \neq 0$ are when $\eta = (n, n-2)$ or $\eta = (n-1, n-1)$ and in both cases $c_{\eta,(1)}^{(n,n-1)} = 1$. Thus

$$\begin{aligned} (s_{(1)}s_{(n-1,n-1)}) * s_{(n,n-1)} &= s_{(1)}(s_{(n,n-2)} * s_{(n-1,n-1)}) \\ &\quad + s_{(1)}(s_{(n-1,n-1)} * s_{(n-1,n-1)}). \end{aligned}$$

If $\theta \vdash 2n-1$, then (4) and the above equation together give

$$\begin{aligned} \langle s_{(n-1,n-1,1)} * s_{(n,n-1)}, s_{\theta} \rangle &= \langle s_{(1)}((s_{(n,n-2)} + s_{(n-1,n-1)}) * s_{(n-1,n-1)}), s_{\theta} \rangle \\ &\quad - \langle s_{(n,n-1)} * s_{(n,n-1)}, s_{\theta} \rangle \\ &= \langle (s_{(n,n-2)} + s_{(n-1,n-1)}) * s_{(n-1,n-1)}, s_{\theta/(1)} \rangle \\ &\quad - \langle s_{(n,n-1)} * s_{(n,n-1)}, s_{\theta} \rangle \\ &= \left\langle \sum_{\substack{\lambda \vdash 2n-2 \\ l(\lambda) \leq 4}} s_{\lambda}, s_{\theta/(1)} \right\rangle - \left\langle \sum_{\substack{\lambda \vdash 2n-1 \\ l(\lambda) \leq 4}} s_{\lambda}, s_{\theta} \right\rangle. \end{aligned} \tag{5}$$

It is straightforward to verify that this gives the same characterization as the one obtained from $s_{(n,n-1,1)} * s_{(n,n)}$ by applying the same argument, except that we use (5) instead of (3). Hence we obtain the following theorem.

Theorem 3.1. *Let $\lambda = (n-1, n-1, 1)$, $\mu = (n, n-1)$ and $\theta \vdash 2n-1$. Then the Kronecker coefficients labelled by these partitions are as follows.*

$$g_{\lambda\mu}^{\theta} = \begin{cases} 1 & l(\theta) = 5, \theta_5 = 1 \\ d_{\theta} - 1 & l(\theta) \leq 4 \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.2. *Theorem 3.1 gives the following expansion for $s_{(3,3,1)} * s_{(4,3)}$.*

$$\begin{aligned} s_{(3,3,1)} * s_{(4,3)} &= s_{(2,2,1,1,1)} + s_{(2,2,2,1)} + s_{(3,1,1,1,1)} + 2s_{(3,2,1,1)} + s_{(3,2,2)} + s_{(3,3,1)} \\ &\quad + s_{(4,1,1,1)} + 2s_{(4,2,1)} + s_{(4,3)} + s_{(5,1,1)} + s_{(5,2)} + s_{(6,1)}. \end{aligned}$$

4. THE KRONECKER COEFFICIENT $g_{(n-1,n-1,2)(n,n)}^{\theta}$ FOR $n \geq 3$

Before we derive the Kronecker coefficients occurring in the product $s_{(n-1,n-1,2)} * s_{(n,n)}$, we will make a remark about our notation. From this section onwards, the statement ' $\lambda \in P$ ' is considered to be equivalent to ' $\lambda' \in P$ ', and an analogous statement holds for a statement of the form ' $\lambda \in Q$ '. For example, consider $\lambda = (5, 3, 3, 1, 1)$. Then, even though λ has 5

parts, we say $((\lambda \in P))$ evaluates to 1 because $\lambda' = (5, 3, 3, 1)$ has all 4 parts odd, and thus belongs to P .

Now we will calculate $s_{(n-1, n-1, 2)} * s_{(n, n)}$. Firstly, the Pieri rule yields

$$s_{(n-1, n-1, 2)} = s_{(2)}s_{(n-1, n-1)} - s_{(n, n-1, 1)} - s_{(n+1, n-1)}. \quad (6)$$

Using Theorem 1.8, we get

$$\begin{aligned} (s_{(2)}s_{(n-1, n-1)}) * s_{(n, n)} &= \sum_{\delta \vdash 2} \sum_{\eta \vdash 2n-2} c_{\eta, \delta}^{(n, n)} (s_{\eta} * s_{(n-1, n-1)}) (s_{\delta} * s_{(2)}) \\ &= \sum_{\eta \vdash 2n-2} c_{\eta, (1, 1)}^{(n, n)} (s_{\eta} * s_{(n-1, n-1)}) (s_{(1, 1)} * s_{(2)}) \\ &\quad + \sum_{\eta \vdash 2n-2} c_{\eta, (2)}^{(n, n)} (s_{\eta} * s_{(n-1, n-1)}) (s_{(2)} * s_{(2)}). \end{aligned}$$

Notice that for $c_{\eta, (2)}^{(n, n)} \neq 0$ to hold, we must have $\eta = (n, n-2)$ whereas $c_{\eta, (1, 1)}^{(n, n)} \neq 0$ implies that $\eta = (n-1, n-1)$. In both cases, $c_{\eta, \delta}^{(n, n)} = 1$ where $\delta = (1, 1)$, $\eta = (n-1, n-1)$ or $\delta = (2)$, $\eta = (n, n-2)$. This allows us to rewrite the above equation as

$$(s_{(2)}s_{(n-1, n-1)}) * s_{(n, n)} = s_{(1, 1)}(s_{(n-1, n-1)} * s_{(n-1, n-1)}) + s_{(2)}(s_{(n, n-2)} * s_{(n-1, n-1)}).$$

Now let $\theta \vdash 2n$. From the equality above and (6), it follows that

$$\begin{aligned} \langle s_{(n-1, n-1, 2)} * s_{(n, n)}, s_{\theta} \rangle &= \langle (s_{(2)}s_{(n-1, n-1)}) * s_{(n, n)}, s_{\theta} \rangle \\ &\quad - \langle s_{(n, n-1, 1)} * s_{(n, n)}, s_{\theta} \rangle - \langle s_{(n+1, n-1)} * s_{(n, n)}, s_{\theta} \rangle \\ &= \langle s_{(1, 1)}(s_{(n-1, n-1)} * s_{(n-1, n-1)}), s_{\theta} \rangle \\ &\quad + \langle s_{(2)}(s_{(n, n-2)} * s_{(n-1, n-1)}), s_{\theta} \rangle \\ &\quad - \langle s_{(n, n-1, 1)} * s_{(n, n)}, s_{\theta} \rangle - \langle s_{(n+1, n-1)} * s_{(n, n)}, s_{\theta} \rangle \\ &= \langle s_{(n-1, n-1)} * s_{(n-1, n-1)}, s_{\theta/(1, 1)} \rangle + \langle s_{(n, n-2)} * s_{(n-1, n-1)}, s_{\theta/(2)} \rangle \\ &\quad - \langle s_{(n+1, n-1)} * s_{(n, n)}, s_{\theta} \rangle - \langle s_{(n, n-1, 1)} * s_{(n, n)}, s_{\theta} \rangle. \end{aligned} \quad (7)$$

Since we already have a description for $s_{(n, n-1, 1)} * s_{(n, n)}$ and $s_{(n+1, n-1)} * s_{(n, n)}$ in Theorems 2.1 and 1.7 respectively, we will focus on evaluating the other two terms on the right hand side of (7). Notice first that, by Theorem 1.10, if $\langle s_{(n-1, n-1, 2)} * s_{(n, n)}, s_{\theta} \rangle \neq 0$ then $l(\theta) \leq 6$ necessarily. So we will restrict ourselves to partitions $\theta \vdash 2n$ satisfying $l(\theta) \leq 6$ and analyse cases based on the length of the partition θ . Before we begin our case analysis, an important remark is necessary.

Remark. We will be using the Pieri rule to compute the expansion of $s_{\theta/(2)}$ and $s_{\theta/(1, 1)}$. Also, we will use Theorem 1.7 for the Kronecker products $s_{(n, n-2)} * s_{(n-1, n-1)}$ and $s_{(n+1, n-1)} * s_{(n, n)}$, Theorem 1.6 for $s_{(n-1, n-1)} * s_{(n-1, n-1)}$, and Theorem 2.1 for $s_{(n, n-1, 1)} * s_{(n, n)}$.

4.1. **Case I: $l(\theta) = 6$.** Clearly, both $\langle s_{(n+1,n-1)} * s_{(n,n)}, s_\theta \rangle$ and $\langle s_{(n,n-1,1)} * s_{(n,n)}, s_\theta \rangle$ are 0. We will compute $\langle s_{(n,n-2)} * s_{(n-1,n-1)}, s_{\theta/(2)} \rangle$. Note that $s_{\theta/(2)}$ is a sum of terms of the form s_δ where either δ is a partition obtained either by subtracting 2 from one of the parts of θ , or by subtracting 1 from 2 distinct parts of θ . In both cases, $l(\delta) \geq 5$. Thus

$$\langle s_{(n,n-2)} * s_{(n-1,n-1)}, s_{\theta/(2)} \rangle = 0.$$

Consider $\langle s_{(n-1,n-1)} * s_{(n-1,n-1)}, s_{\theta/(1,1)} \rangle$ next. Now, $s_{\theta/(1,1)}$ is a sum of terms of the form s_δ , where $\delta \vdash 2n-2$ is obtained by subtracting 1 each from two different (but not necessarily distinct) parts of θ . Thus, we obtain the following

$$\langle s_{(n-1,n-1)} * s_{(n-1,n-1)}, s_{\theta/(1,1)} \rangle = \begin{cases} 1 & \theta \in P, \theta_5 = \theta_6 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Hence, in the present case, (7) gives

$$\langle s_{(n-1,n-1,2)} * s_{(n,n)}, s_\theta \rangle = \begin{cases} 1 & \theta_5 = \theta_6 = 1, \theta \in P \\ 0 & \text{otherwise.} \end{cases}$$

4.2. **Case II: $l(\theta) = 5$.** In this case, we know that $\langle s_{(n+1,n-1)} * s_{(n,n)}, s_\theta \rangle = 0$ and

$$\langle s_{(n,n-1,1)} * s_{(n,n)}, s_\theta \rangle = \begin{cases} 1 & \theta_5 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

We will compute $\langle s_{(n,n-2)} * s_{(n-1,n-1)}, s_{\theta/(2)} \rangle$ next. Firstly, note that if $\theta_5 \geq 3$, then $s_{\theta/(2)}$ is a sum of terms of the form s_δ where $l(\gamma) = 5$ implying that $\langle s_{(n,n-2)} * s_{(n-1,n-1)}, s_{\theta/(2)} \rangle = 0$. If $\theta_5 = 2$, then $s_{\theta/(2)} = s_{\theta'} +$ sum of terms of the form s_δ where $l(\delta) = 5$. Thus, $\langle s_{(n,n-2)} * s_{(n-1,n-1)}, s_{\theta/(2)} \rangle = 1$ if $\theta \in Q$.

The case where $\theta_5 = 1$ is more intricate. Consider first the case where $\theta_4 = \theta_5 = 1$. Note that since $\theta' \vdash 2n-1$, we know that, for $1 \leq i \leq 3$, either all θ_i are even or exactly 2 are odd. Note also that $s_{(n,n-1)} * s_{(n,n)}$ is a sum of terms of the form s_γ where $l(\gamma) \leq 4$ and $\gamma \in Q$. Thus, to compute $\langle s_{(n,n-1)} * s_{(n-1,n-1)}, s_{\theta/(2)} \rangle$, we only need to focus on those terms s_δ in $s_{\theta/(2)}$ that satisfy $l(\delta) \leq 4$ and $\delta \in Q$. Such a partition δ can only be obtained by removing θ_5 and then subtracting 1 from some $\theta_i \geq 2$ for $1 \leq i \leq 3$. Analyzing when such a process gives δ belonging to Q implies that

$$\langle s_{(n,n-2)} * s_{(n-1,n-1)}, s_{\theta/(2)} \rangle = ((E_\theta = 1))[O'_\theta - 1] + ((O_\theta = 1))[E'_\theta].$$

Now consider the case where $\theta_5 = 1$ but $\theta_4 \geq 2$. Note that either exactly 3 parts in θ' are odd, or exactly 3 parts are even. As was the case earlier, we only need to focus on those terms s_δ in $s_{\theta/(2)}$ that satisfy $l(\delta) \leq 4$ and $\delta \in Q$. Such a partition δ can only be obtained by removing θ_5 and then subtracting 1 from some θ_i for $1 \leq i \leq 4$. Analyzing when such a process gives δ belonging to Q implies that

$$\langle s_{(n,n-2)} * s_{(n-1,n-1)}, s_{\theta/(2)} \rangle = ((E_\theta = 1))[O'_\theta] + ((O_\theta = 1))[E'_\theta].$$

Collecting the above results, we obtain that

$$\langle s_{(n,n-2)} * s_{(n-1,n-1)}, s_{\theta/(2)} \rangle = \begin{cases} 0 & \theta_5 \geq 3 \\ 1 & \theta_5 = 2, \theta \in Q \\ O'_\theta - ((\theta_4 = 1)) & \theta_5 = 1, E_\theta = 1 \\ E'_\theta & \theta_5 = 1, O_\theta = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Next we compute $\langle s_{(n-1,n-1)} * s_{(n-1,n-1)}, s_{\theta/(1,1)} \rangle$. Observe that if $\theta_5 \geq 2$, then $s_{\theta/(1,1)}$ is a sum of terms of the form s_δ with $l(\delta) = 5$ and these terms do not appear in $s_{(n-1,n-1)} * s_{(n-1,n-1)}$. This allows us to narrow our consideration to the case $\theta_5 = 1$. Since θ' is a partition of $2n - 1$, it has either exactly three parts even, or exactly 3 parts odd. We also have that $s_{\theta/(1,1)}$ is a sum of terms of the form s_δ , where either $l(\delta) = 5$, in which case they do not occur in $s_{(n-1,n-1)} * s_{(n-1,n-1)}$, or $l(\delta) \leq 4$. Clearly, the only way to obtain a partition δ satisfying $l(\delta) \leq 4$ is to remove θ_5 and subtract 1 from some part θ_i for $1 \leq i \leq 4$. Careful analysis shows there is exactly one partition $\delta \in P$ that can be obtained this way. This implies that

$$\langle s_{(n-1,n-1)} * s_{(n-1,n-1)}, s_{\theta/(1,1)} \rangle = \begin{cases} 1 & \theta_5 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Collecting the results allows us to rewrite (7) in the case $l(\theta) = 5$ as follows.

$$\langle s_{(n-1,n-1,2)} * s_{(n,n)}, s_\theta \rangle = \begin{cases} 1 & \theta_5 = 2, \theta \in Q \\ O'_\theta - ((\theta_4 = 1)) & E_\theta = 1, \theta_5 = 1 \\ E'_\theta & O_\theta = 1, \theta_5 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

4.3. Case III: $l(\theta) \leq 4$. We know that, if $l(\theta) \leq 4$, then

$$\begin{aligned} \langle s_{(n+1,n-1)} * s_{(n,n)}, s_\theta \rangle &= ((\theta \in Q)) \\ \langle s_{(n,n-1,1)} * s_{(n,n)}, s_\theta \rangle &= d_\theta - 1. \end{aligned}$$

Consider $\langle s_{(n,n-2)} * s_{(n-1,n-1)}, s_{\theta/(2)} \rangle$ with $\theta \in P$. A partition obtained by subtracting 2 from any part of θ is still in P , and there are no terms of the form s_γ with $\gamma \in P$ in $s_{(n,n-2)} * s_{(n-1,n-1)}$. Thus the only partitions δ such that s_δ appears in $s_{\theta/(2)}$, and that contribute to the inner product are obtained by subtracting 1 from two distinct parts of θ . Furthermore, any partition δ so obtained clearly belongs to Q .

If $\theta \in Q$, then all partitions obtained by subtracting 2 from a part of θ belong to Q . To obtain other partitions δ such that $\delta \in Q$ and s_δ appears in $s_{\theta/(2)}$, we subtract 1 from one of the odd parts and 1 from one of the even parts. Thus, we obtain

$$\langle s_{(n,n-2)} * s_{(n-1,n-1)}, s_{\theta/(2)} \rangle = \begin{cases} \binom{d_\theta}{2} & \theta \in P \\ d_{\theta,2} + O'_\theta E'_\theta & \theta \in Q. \end{cases}$$

Finally, consider $\langle s_{(n-1,n-1)} * s_{(n-1,n-1)}, s_{\theta/(1,1)} \rangle$. Assume first that $\theta \in P$. Then notice that $s_{\theta/(1,1)}$ is a sum of terms of the form s_δ where $\delta \in Q$. This implies that $\langle s_{(n-1,n-1)} * s_{(n-1,n-1)}, s_{\theta/(1,1)} \rangle = 0$. The remaining case is $\theta \in Q$. Thus, there are exactly 2 odd parts in θ . Subtracting 1 from each of these parts will give us a partition of $2n - 2$ that lies in P . If there are 2 even parts in θ , then subtracting 1 from each of these will also give us a partition of $2n - 2$ lying in P . Thus, if $l(\theta) \leq 4$,

$$\langle s_{(n-1,n-1)} * s_{(n-1,n-1)}, s_{\theta/(1,1)} \rangle = \begin{cases} 0 & \theta \in P \\ 1 + ((E_\theta = 2)) & \theta \in Q. \end{cases}$$

We are now ready to state a formula for the Kronecker coefficient $g_{(n-1,n-1,2)(n,n)}^\theta$ where $\theta \vdash 2n$.

Theorem 4.1. *Let $\lambda = (n - 1, n - 1, 2)$, $\mu = (n, n)$ and $\theta \vdash 2n$. Then the Kronecker coefficients labelled by these partitions are as follows.*

$$g_{\lambda\mu}^\theta = \begin{cases} ((\theta \in P)) & l(\theta) = 6 \text{ and } \theta_5 = \theta_6 = 1 \\ ((\theta \in Q)) & l(\theta) = 5, \theta_5 = 2 \\ O'_\theta - ((\theta_4 = 1)) & l(\theta) = 5, E_\theta = 1 \text{ and } \theta_5 = 1 \\ E'_\theta & l(\theta) = 5, O_\theta = 1 \text{ and } \theta_5 = 1 \\ 1 - d_\theta + \binom{d_\theta}{2} & l(\theta) \leq 4, \theta \in P \\ 1 - d_\theta + d_{\theta,2} + O'_\theta E'_\theta + ((E_\theta = 2)) & l(\theta) \leq 4, \theta \in Q \\ 0 & \text{otherwise.} \end{cases}$$

Example 4.2. *Let $\lambda = (7, 7, 2)$ and $\mu = (8, 8)$. We will use the above characterization to compute the coefficients of $s_{(5,5,3,1,1,1)}$, $s_{(6,4,3,2,1)}$ and $s_{(7,5,2,2)}$ in the Kronecker product $s_\lambda * s_\mu$. For the sake of convenience let $\alpha = (5, 5, 3, 1, 1, 1)$, $\beta = (6, 4, 3, 2, 1)$ and $\gamma = (7, 5, 2, 2)$.*

To compute $g_{\lambda\mu}^\alpha$, notice that as $l(\alpha) = 6$ and $\alpha_5 = \alpha_6 = 1$, Theorem 4.1 states that

$$g_{\lambda\mu}^\alpha = ((\alpha \in P)).$$

Since $\alpha' = (5, 5, 3, 1) \in P$, we obtain

$$g_{(7,7,2)(8,8)}^{(5,5,3,1,1,1)} = 1.$$

To compute $g_{\lambda\mu}^\beta$, notice first that $l(\beta) = 5$ and $\beta_5 = 1$. Furthermore, as β' has exactly 1 odd part, we have $O_\beta = 1$. Thus, Theorem 4.1 states that

$$g_{\lambda\mu}^\beta = E'_\beta.$$

Since $\beta' = (6, 4, 3, 2)$ has exactly 3 distinct even parts, we have

$$g_{(7,7,2)(8,8)}^{(6,4,3,2,1)} = 3.$$

To compute $g_{\lambda\mu}^\gamma$, notice that $l(\gamma) = 4$ and $\gamma \in Q$. Theorem 4.1 states that

$$g_{\lambda\mu}^\gamma = 1 - d_\gamma + d_{\gamma,2} + O'_\gamma E'_\gamma + ((E_\gamma = 2)).$$

We have $d_\gamma = 3$, $d_{\gamma,2} = 3$, $O'_\gamma = 2$, $E'_\gamma = 1$ and $E_\gamma = 2$. Thus, we get

$$g_{(7,7,2)(8,8)}^{(7,5,2,2)} = 1 - 3 + 3 + 2 + 1 = 4.$$

5. THE KRONECKER COEFFICIENT $g_{(n-1,n-1,1,1)(n,n)}^\theta$ FOR $n \geq 2$

The derivation of the coefficients in this section is similar to that in the previous sections, and we begin again with the Pieri rule. It implies that

$$s_{(n-1,n-1,1,1)} = s_{(1,1)}s_{(n-1,n-1)} - s_{(n,n)} - s_{(n,n-1,1)}. \quad (8)$$

Using Theorem 1.8, we deduce that

$$\begin{aligned} (s_{(1,1)}s_{(n-1,n-1)}) * s_{(n,n)} &= \sum_{\delta \vdash 2} \sum_{\eta \vdash 2n-2} c_{\eta,\delta}^{(n,n)} (s_\eta * s_{(n-1,n-1)})(s_\delta * s_{(1,1)}) \\ &= \sum_{\eta \vdash 2n-2} c_{\eta,(1,1)}^{(n,n)} (s_\eta * s_{(n-1,n-1)})(s_{(1,1)} * s_{(1,1)}) \\ &\quad + \sum_{\eta \vdash 2n-2} c_{\eta,(2)}^{(n,n)} (s_\eta * s_{(n-1,n-1)})(s_{(2)} * s_{(1,1)}). \end{aligned}$$

We have already computed those η that satisfy $c_{\eta,\delta}^{(n,n)} \neq 0$ for $\delta \vdash 2$ in the previous section. Since $s_{(1,1)} * s_{(1,1)} = s_{(2)}$, we obtain the following.

$$(s_{(1,1)}s_{(n-1,n-1)}) * s_{(n,n)} = s_{(2)}(s_{(n-1,n-1)} * s_{(n-1,n-1)}) + s_{(1,1)}(s_{(n,n-2)} * s_{(n-1,n-1)})$$

Using (8) and the equality above, and given $\theta \vdash 2n$, we obtain

$$\begin{aligned} \langle s_{(n-1,n-1,1,1)} * s_{(n,n)}, s_\theta \rangle &= \langle (s_{(1,1)}s_{(n-1,n-1)}) * s_{(n,n)}, s_\theta \rangle \\ &\quad - \langle s_{(n,n-1,1)} * s_{(n,n)}, s_\theta \rangle - \langle s_{(n,n)} * s_{(n,n)}, s_\theta \rangle \\ &= \langle s_{(2)}(s_{(n-1,n-1)} * s_{(n-1,n-1)}), s_\theta \rangle \\ &\quad + \langle s_{(1,1)}(s_{(n,n-2)} * s_{(n-1,n-1)}), s_\theta \rangle \\ &\quad - \langle s_{(n,n-1,1)} * s_{(n,n)}, s_\theta \rangle - \langle s_{(n,n)} * s_{(n,n)}, s_\theta \rangle \\ &= \langle s_{(n-1,n-1)} * s_{(n-1,n-1)}, s_{\theta/(2)} \rangle \\ &\quad + \langle s_{(n,n-2)} * s_{(n-1,n-1)}, s_{\theta/(1,1)} \rangle \\ &\quad - \langle s_{(n,n-1,1)} * s_{(n,n)}, s_\theta \rangle - \langle s_{(n,n)} * s_{(n,n)}, s_\theta \rangle. \quad (9) \end{aligned}$$

As we have done in the earlier sections, we proceed to evaluate individual terms on the right hand side of (9). Also, with the results we obtained in the earlier sections it only remains to compute $\langle s_{(n-1,n-1)} * s_{(n-1,n-1)}, s_{\theta/(2)} \rangle$ and $\langle s_{(n,n-2)} * s_{(n-1,n-1)}, s_{\theta/(1,1)} \rangle$.

Remark. We will be using the Pieri rule to compute the expansion of $s_{\theta/(2)}$ and $s_{\theta/(1,1)}$. Also, we will use Theorem 1.7 for the Kronecker products $s_{(n,n-2)} * s_{(n-1,n-1)}$ and $s_{(n+1,n-1)} * s_{(n,n)}$, Theorem 1.6 for $s_{(n-1,n-1)} * s_{(n-1,n-1)}$, and Theorem 2.1 for $s_{(n,n-1,1)} * s_{(n,n)}$.

5.1. **Case I: $l(\theta) = 6$.** Clearly both $\langle s_{(n,n)} * s_{(n,n)}, s_\theta \rangle$ and $\langle s_{(n,n-1,1)} * s_{(n,n)}, s_\theta \rangle$ are 0 in this case. Consider $\langle s_{(n-1,n-1)} * s_{(n-1,n-1)}, s_{\theta/(2)} \rangle$ next. This is 0 if $l(\theta) \geq 6$ as $s_{\theta/(2)}$ is a sum of terms of the form s_δ with $l(\delta) \geq 5$.

Now consider $\langle s_{(n,n-2)} * s_{(n-1,n-1)}, s_{\theta/(1,1)} \rangle$. The only possibility for a non-zero coefficient is when $\theta_5 = \theta_6 = 1$. Thus

$$\langle s_{(n,n-2)} * s_{(n-1,n-1)}, s_{\theta/(1,1)} \rangle = \begin{cases} 1 & \theta_5 = \theta_6 = 1, \theta \in Q \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, (9) reduces to

$$\langle s_{(n-1,n-1,1,1)} * s_{(n,n)}, s_\theta \rangle = \begin{cases} 1 & \theta_5 = \theta_6 = 1, \theta \in Q \\ 0 & \text{otherwise.} \end{cases}$$

5.2. **Case II: $l(\theta) = 5$.** In this case we have $\langle s_{(n,n)} * s_{(n,n)}, s_\theta \rangle = 0$ and

$$\langle s_{(n,n-1,1)} * s_{(n,n)}, s_\theta \rangle = \begin{cases} 1 & \theta_5 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Consider $\langle s_{(n-1,n-1)} * s_{(n-1,n-1)}, s_{\theta/(2)} \rangle$ now. If $l(\theta) = 5$ and $\theta_5 \geq 3$, this is 0 because $s_{\theta/(2)}$ is a sum of terms of the form s_δ with $l(\delta) = 5$.

If $\theta_5 = 2$, then the only term in $s_{\theta/(2)}$ that is of the form s_δ with $l(\delta) \leq 4$ is s_θ . Thus, one obtains

$$\langle s_{(n-1,n-1)} * s_{(n-1,n-1)}, s_{\theta/(2)} \rangle = \begin{cases} 1 & \theta \in P \\ 0 & \text{otherwise.} \end{cases}$$

Next consider $\theta_5 = 1$. We are interested in partitions δ that satisfy $l(\delta) \leq 4$ and $\delta \in P$, and furthermore, are such that s_δ appears in $s_{\theta/(2)}$. Notice that θ' has either exactly 1 odd part or exactly 1 even part. Thus, to obtain a partition satisfying the conditions above, we need to remove θ_5 and then subtract 1 from an even part if there is exactly one even part or subtract 1 from an odd part if there is exactly one odd part (provided that this odd part does not equal 1). Thus, we obtain

$$\langle s_{(n-1,n-1)} * s_{(n-1,n-1)}, s_{\theta/(2)} \rangle = \begin{cases} 1 & E_\theta = 1 \\ 1 - ((\theta_4 = 1)) & O_\theta = 1. \end{cases}$$

Combining these facts implies that if $l(\theta) = 5$ then

$$\langle s_{(n-1,n-1)} * s_{(n-1,n-1)}, s_{\theta/(2)} \rangle = \begin{cases} 1 & \theta_5 = 2, \theta \in P \\ 1 & \theta_5 = 1, E_\theta = 1 \\ 1 - ((\theta_4 = 1)) & \theta_5 = 1, O_\theta = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Now we will analyze $\langle s_{(n,n-2)} * s_{(n-1,n-1)}, s_{\theta/(1,1)} \rangle$. If $\theta_5 \geq 2$, then this is clearly 0, as $s_{\theta/(1,1)}$ consists of terms of the form s_δ with $l(\delta) \geq 5$. Thus, assume that $\theta_5 = 1$. The only way to get a term in $s_{\theta/(1,1)}$ of the form s_δ with $l(\delta) \leq 4$ is to remove θ_5 and subtract 1 from one of the other parts in θ . We further require that δ belongs to Q if s_δ is to have a non-zero coefficient in $s_{(n,n-2)} * s_{(n-1,n-1)}$. The only way to achieve this is to subtract 1 from one of

the odd parts if there is exactly 1 even part in θ' , or subtract 1 from one of the even parts if there is exactly 1 odd part in θ' . Hence, if $l(\theta) = 5$, then

$$\langle s_{(n,n-2)} * s_{(n-1,n-1)}, s_{\theta/(1,1)} \rangle = \begin{cases} E'_\theta & \theta_5 = 1, O_\theta = 1 \\ O'_\theta & \theta_5 = 1, E_\theta = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Using (9), if $l(\theta) = 5$, then we obtain

$$\langle s_{(n-1,n-1,1,1)} * s_{(n,n)}, s_\theta \rangle = \begin{cases} 1 & \theta_5 = 2, \theta \in P \\ O'_\theta & \theta_5 = 1, E_\theta = 1 \\ E'_\theta - ((\theta_4 = 1)) & \theta_5 = 1, O_\theta = 1 \\ 0 & \text{otherwise.} \end{cases}$$

5.3. **Case III: $l(\theta) \leq 4$.** In this case, we have

$$\begin{aligned} \langle s_{(n,n)} * s_{(n,n)}, s_\theta \rangle &= ((\theta \in P)), \\ \langle s_{(n,n-1,1)} * s_{(n,n)}, s_\theta \rangle &= d_\theta - 1. \end{aligned}$$

We will compute $\langle s_{(n-1,n-1)} * s_{(n-1,n-1)}, s_{\theta/(2)} \rangle$ first. If $\theta \in P$, then the only terms in $s_{\theta/(2)}$ which can appear in $s_{(n-1,n-1)} * s_{(n-1,n-1)}$ are of the form s_δ where δ is partition obtained by subtracting 2 from a part of θ , and so $\delta \in P$. Subtracting 1 from two different parts of θ gives a partition in Q , and hence there is no contribution to the aforementioned inner product.

If, on the other hand, $\theta \in Q$, then to get a term of the form s_δ in $s_{\theta/(2)}$ such that $\delta \in P$, the only possibility is to either subtract 1 each from two distinct even parts of θ , or subtract 1 each from two distinct odd parts of θ . These arguments imply

$$\langle s_{(n-1,n-1)} * s_{(n-1,n-1)}, s_{\theta/(2)} \rangle = \begin{cases} d_{\theta,2} & \theta \in P \\ ((E'_\theta = 2)) + ((O'_\theta = 2)) & \theta \in Q. \end{cases}$$

Now consider $\langle s_{(n,n-2)} * s_{(n-1,n-1)}, s_{\theta/(1,1)} \rangle$. If $\theta \in P$, then subtracting 1 each from any two parts in θ gives a partition in Q , if what is obtained after the subtraction is indeed a partition. If $\theta \in Q$, then subtracting 1 from one of the even parts and subtracting 1 from one of the odd parts gives a partition in Q . Thus, we conclude that

$$\langle s_{(n,n-2)} * s_{(n-1,n-1)}, s_{\theta/(1,1)} \rangle = \begin{cases} \binom{d_\theta}{2} + R_\theta & \theta \in P \\ O'_\theta E'_\theta & \theta \in Q. \end{cases}$$

Having analyzed all cases, we will now give a formula for $g_{(n-1,n-1,1,1)(n,n)}^\theta$, where $\theta \vdash 2n$, in the following theorem.

Theorem 5.1. *Let $\lambda = (n - 1, n - 1, 1, 1)$, $\mu = (n, n)$ and $\theta \vdash 2n$. Then the Kronecker coefficients labelled by these partitions are as follows.*

$$g_{\lambda\mu}^{\theta} = \begin{cases} ((\theta \in Q)) & l(\theta) = 6 \text{ and } \theta_5 = \theta_6 = 1 \\ 1 & l(\theta) = 5, \theta_5 = 2 \text{ and } \theta \in P \\ O'_{\theta} & l(\theta) = 5, E_{\theta} = 1 \text{ and } \theta_5 = 1 \\ E'_{\theta} - ((\theta_4 = 1)) & l(\theta) = 5, O_{\theta} = 1 \text{ and } \theta_5 = 1 \\ d_{\theta,2} - d_{\theta} + \binom{d_{\theta}}{2} + R_{\theta} & l(\theta) \leq 4, \theta \in P \\ 1 - d_{\theta} + ((E'_{\theta} = 2)) + ((O'_{\theta} = 2)) + O'_{\theta}E'_{\theta} & l(\theta) \leq 4, \theta \in Q \\ 0 & \text{otherwise.} \end{cases}$$

Example 5.2. *Let $\lambda = (7, 7, 1, 1)$ and $\mu = (8, 8)$. We will use Theorem 5.1 to compute the coefficients of $s_{(5,5,3,1,1,1)}$, $s_{(6,4,3,2,1)}$ and $s_{(7,5,2,2)}$ in the Kronecker product $s_{\lambda} * s_{\mu}$. For readability, let $\alpha = (5, 5, 3, 1, 1, 1)$, $\beta = (6, 4, 3, 2, 1)$ and $\gamma = (7, 5, 2, 2)$.*

To compute $g_{\lambda\mu}^{\alpha}$, note that $l(\alpha) = 6$ and $\alpha_5 = \alpha_6 = 1$. Thus the above characterization implies that

$$g_{\lambda\mu}^{\alpha} = ((\alpha \in Q)).$$

Since $\alpha' = (5, 5, 3, 1) \notin Q$, we obtain

$$g_{(7,7,1,1)(8,8)}^{(5,5,3,1,1,1)} = 0.$$

To compute $g_{\lambda\mu}^{\beta}$, note that we have $l(\beta) = 5$ and $\beta_5 = 1$. Since $\beta' = (6, 4, 3, 2)$, we also have $O_{\beta} = 1$. Theorem 5.1 states that

$$g_{\lambda\mu}^{\beta} = E'_{\beta} - ((\beta_4 = 1)).$$

Since β' has 3 distinct even parts and $\beta_4 \neq 1$, we obtain

$$g_{(7,7,1,1)(8,8)}^{(6,4,3,2,1)} = 3.$$

To compute $g_{\lambda\mu}^{\gamma}$, note that $l(\gamma) = 4$ and $\gamma \in Q$. Thus, Theorem 5.1 implies that

$$g_{\lambda\mu}^{\gamma} = 1 - d_{\gamma} + ((E'_{\gamma} = 2)) + ((O'_{\gamma} = 2)) + O'_{\gamma}E'_{\gamma}.$$

We have $d_{\gamma} = 3$, $O'_{\gamma} = 2$ and $E'_{\gamma} = 1$. Thus

$$g_{(7,7,1,1)(8,8)}^{(7,5,2,2)} = 1 - 3 + 0 + 1 + 2 = 1.$$

6. THE KRONECKER COEFFICIENT $g_{(n,n,1)(n,n,1)}^{\theta}$ FOR $n \geq 2$

In this section we will compute the Kronecker product $s_{(n,n,1)} * s_{(n,n,1)}$. Before commencing our calculations, we need to introduce certain statistics on partitions. These will allow us to deduce the relation between the number of distinct parts in a partition θ and in a partition $\theta^- \prec \theta$.

Fix an alphabet $X = \{0, 1, 2\}$. We will associate a string σ of length $l(\theta) + 1$ to a partition θ . For $1 \leq i \leq l(\theta)$, define

$$\sigma_i = \begin{cases} 0 & (\theta_i = \theta_{i+1}) \\ 1 & (\theta_i - \theta_{i+1} = 1) \\ 2 & (\theta_i - \theta_{i+1} \geq 2). \end{cases}$$

Here we are assuming that when $i = l(\theta)$, then $\theta_{i+1} = 0$. Also, define $\sigma_0 = \sigma_1$. Having computed σ , define the following sets.

$$\begin{aligned} A_{\theta,1} &= \{i : 1 \leq i \leq l(\theta), \sigma_i = 1 \text{ and } \sigma_{i-1} = 0\} \\ A_{\theta,2} &= \{i : 1 \leq i \leq l(\theta), \sigma_i = 2 \text{ and } \sigma_{i-1} = 0\} \\ B_{\theta,1} &= \{i : 1 \leq i \leq l(\theta), \sigma_i = 1 \text{ and } \sigma_{i-1} \neq 0\} \\ B_{\theta,2} &= \{i : 1 \leq i \leq l(\theta), \sigma_i = 2 \text{ and } \sigma_{i-1} \neq 0\} \end{aligned}$$

Note that to obtain θ^- from θ , one can subtract 1 only from those parts θ_i such that $\sigma_i = 1$ or 2. If θ^- is obtained by subtracting 1 from θ_i where $i \in A_{\theta,1}$ or $i \in B_{\theta,2}$, then $d_{\theta^-} = d_{\theta}$. For $i \in A_{\theta,2}$, subtracting 1 from θ_i results in d_{θ^-} being $d_{\theta} + 1$. Finally, for $i \in B_{\theta,1}$, subtracting 1 from θ_i results in d_{θ^-} being $d_{\theta} - 1$. We will use $a_{\theta,1}$, $a_{\theta,2}$, $b_{\theta,1}$ and $b_{\theta,2}$ to denote the cardinalities of the sets $A_{\theta,1}$, $A_{\theta,2}$, $B_{\theta,1}$ and $B_{\theta,2}$ respectively.

Example 6.1. Consider the partition $\theta = (8, 6, 2, 1)$. Then the string σ associated with θ will be of length 5, as $l(\theta) = 4$. Since $\theta_1 - \theta_2$ and $\theta_2 - \theta_3$ are both ≥ 2 , we have that $\sigma_1 = \sigma_2 = 2$. Since $\theta_3 - \theta_4 = 1$, we get that $\sigma_3 = 1$. Recall now that we are assuming $\theta_5 = 0$. This implies that $\sigma_4 = 1$ as $\theta_4 - \theta_5$ equals 1. Finally, we have that $\sigma_0 = \sigma_1 = 2$ by definition. Hence,

$$\sigma = 22211.$$

For this partition θ , it is easy to see that $A_{\theta,1}$ and $A_{\theta,2}$ are empty sets, while

$$B_{\theta,1} = \{3, 4\}, \quad B_{\theta,2} = \{1, 2\}.$$

Thus, we have that $a_{\theta,1} = a_{\theta,2} = 0$ while $b_{\theta,1} = b_{\theta,2} = 2$.

Now we will begin the calculations required to compute $s_{(n,n,1)} * s_{(n,n,1)}$. The Pieri rule implies that

$$s_{(1)}s_{(n,n)} = s_{(n,n,1)} + s_{(n+1,n)}.$$

This yields that

$$\begin{aligned} s_{(n,n,1)} * s_{(n,n,1)} &= (s_{(1)}s_{(n,n)} - s_{(n+1,n)}) * s_{(n,n,1)} \\ &= (s_{(1)}s_{(n,n)}) * s_{(n,n,1)} - s_{(n+1,n)} * s_{(n,n,1)}. \end{aligned} \tag{10}$$

Now, Theorem 1.8 gives

$$\begin{aligned} (s_{(1)}s_{(n,n)}) * s_{(n,n,1)} &= \sum_{\delta \vdash 1} \sum_{\eta \vdash 2n} c_{\eta, \delta}^{(n,n,1)} (s_{\eta} * s_{(n,n)}) (s_{\delta} * s_{(1)}) \\ &= \sum_{\eta \vdash 2n} c_{\eta, (1)}^{(n,n,1)} (s_{\eta} * s_{(n,n)}) (s_{(1)} * s_{(1)}). \end{aligned} \quad (11)$$

Next we need to compute which partitions $\eta \vdash 2n$ give a non-zero value for $c_{\eta, (1)}^{(n,n,1)}$. The Pieri rule implies that $c_{\eta, (1)}^{(n,n,1)} = 0$ for all η except $\eta = (n, n)$ and $\eta = (n, n-1, 1)$. It implies further that $c_{(n,n), (1)}^{(n,n,1)} = c_{(n, n-1, 1), (1)}^{(n,n,1)} = 1$. Using the fact that $s_{(1)} * s_{(1)} = s_{(1)}$, (11) becomes

$$(s_{(1)}s_{(n,n)}) * s_{(n,n,1)} = s_{(1)} (s_{(n,n)} * s_{(n,n)}) + s_{(1)} (s_{(n,n)} * s_{(n, n-1, 1)}),$$

and using this in (10) gives us

$$s_{(n,n,1)} * s_{(n,n,1)} = s_{(1)} (s_{(n,n)} * s_{(n,n)} + s_{(n,n)} * s_{(n, n-1, 1)}) - s_{(n+1, n)} * s_{(n,n,1)}.$$

We are interested in calculating $\langle s_{(n,n,1)} * s_{(n,n,1)}, s_{\theta} \rangle$, where $\theta \vdash 2n+1$. To this end, the equation above implies that

$$\begin{aligned} \langle s_{(n,n,1)} * s_{(n,n,1)}, s_{\theta} \rangle &= \langle s_{(1)} (s_{(n,n)} * s_{(n,n)} + s_{(n,n)} * s_{(n, n-1, 1)}), s_{\theta} \rangle \\ &\quad - \langle s_{(n+1, n)} * s_{(n,n,1)}, s_{\theta} \rangle \\ &= \langle s_{(n,n)} * s_{(n,n)}, s_{\theta/(1)} \rangle + \langle s_{(n,n)} * s_{(n, n-1, 1)}, s_{\theta/(1)} \rangle \\ &\quad - \langle s_{(n+1, n)} * s_{(n,n,1)}, s_{\theta} \rangle. \end{aligned} \quad (12)$$

We will calculate the terms on the right hand side individually. It is clear via Theorem 1.10 that if s_{θ} has a non-zero coefficient in $s_{(n,n,1)} * s_{(n,n,1)}$, then $l(\theta) \leq 6$. We will again proceed using case analysis.

Remark. We will be using the Pieri rule to compute $s_{\theta/(1)}$. Also, we will use Theorem 1.6 for $s_{(n,n)} * s_{(n,n)}$, Theorem 2.1 for $s_{(n, n-1, 1)} * s_{(n,n)}$, and Theorem 3.1 for $s_{(n,n,1)} * s_{(n+1, n)}$.

6.1. Case I: $l(\theta) = 6$. From our description of $s_{(n,n,1)} * s_{(n+1, n)}$, we know that $\langle s_{(n+1, n)} * s_{(n,n,1)}, s_{\theta} \rangle = 0$.

Now, recall that $s_{(n,n)} * s_{(n, n-1, 1)}$ is a sum of terms of the form s_{γ} and $l(\gamma) \leq 5$ for each such term. Also, if $l(\gamma) = 5$ then s_{γ} appears with coefficient 1 if and only if $\gamma_5 = 1$, otherwise the coefficient is 0. This implies that $\langle s_{(n,n)} * s_{(n, n-1, 1)}, s_{\theta/(1)} \rangle = 1$ if and only if $\theta_6 = \theta_5 = 1$, and 0 otherwise.

Finally, since $s_{(n,n)} * s_{(n,n)}$ is a sum of terms of the form s_{γ} where $l(\gamma) \leq 4$, it is clear that $\langle s_{(n,n)} * s_{(n,n)}, s_{\theta/(1)} \rangle = 0$. Thus, if $l(\theta) = 6$, (12) reduces to the following.

$$\langle s_{(n,n,1)} * s_{(n,n,1)}, s_{\theta} \rangle = \begin{cases} 1 & \theta_6 = \theta_5 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

6.2. **Case II: $l(\theta) = 5$.** In this case, we know that $\langle s_{(n+1,n)} * s_{(n,n,1)}, s_\theta \rangle$ is 1 if $\theta_5 = 1$ and 0 otherwise.

Next we will compute $\langle s_{(n,n-1,1)} * s_{(n,n)}, s_{\theta/(1)} \rangle$. Consider the case where $\theta_5 \geq 3$. Then $s_{\theta/(1)}$ is a sum of terms of the form s_δ where $\delta_5 \geq 2$ and these terms do not appear in $s_{(n,n)} * s_{(n,n-1,1)}$. On considering $\theta_5 = 2$, we see that $s_{\theta/(1)}$ has terms of the form s_δ with $\delta_5 = 2$ (which do not appear in $s_{(n,n)} * s_{(n,n-1,1)}$), and exactly one term with $\delta_5 = 1$ which appears in $s_{(n,n)} * s_{(n,n-1,1)}$ with coefficient 1.

The one remaining sub-case here is $\theta_5 = 1$. Since

$$s_{\theta/(1)} = s_{\theta'} + \sum_{\substack{\delta \vdash 2n, \delta_5=1 \\ \langle s_{\theta/(1)}, s_\delta \rangle \neq 0}} s_\delta,$$

we get that

$$\begin{aligned} \langle s_{(n,n)} * s_{(n,n-1,1)}, s_{\theta/(1)} \rangle &= \langle s_{(n,n)} * s_{(n,n-1,1)}, s_{\theta'} \rangle \\ &\quad + \langle s_{(n,n)} * s_{(n,n-1,1)}, \sum_{\substack{\delta \vdash 2n, \delta_5=1 \\ \langle s_{\theta/(1)}, s_\delta \rangle \neq 0}} s_\delta \rangle. \end{aligned} \quad (13)$$

Now we will consider terms on the right hand side of (13) individually. We know that

$$\langle s_{(n,n)} * s_{(n,n-1,1)}, s_{\theta'} \rangle = d_{\theta'} - 1.$$

Notice that $d_{\theta'} - 1$ is related to d_θ in the following manner.

$$d_{\theta'} - 1 = \begin{cases} d_\theta - 2 & \theta_4 \geq 2 \\ d_\theta - 1 & \theta_4 = 1. \end{cases} \quad (14)$$

Observe further that every term s_δ in $s_{\theta/(1)}$ other than $s_{\theta'}$ occurs with coefficient 1 and there are $d_\theta - 1$ such terms. This implies that

$$\langle s_{(n,n)} * s_{(n,n-1,1)}, \sum_{\substack{\delta \vdash 2n, \delta_5=1 \\ \langle s_{\theta/(1)}, s_\delta \rangle \neq 0}} s_\delta \rangle = d_\theta - 1. \quad (15)$$

Using (14) and (15) in (13) gives

$$\langle s_{(n,n-1,1)} * s_{(n,n)}, s_{\theta/(1)} \rangle = \begin{cases} 2d_\theta - 3 & \theta_5 = 1, \theta_4 \geq 2 \\ 2d_\theta - 2 & \theta_5 = 1, \theta_4 = 1. \end{cases}$$

Now consider $\langle s_{(n,n)} * s_{(n,n)}, s_{\theta/(1)} \rangle$. Since $s_{(n,n)} * s_{(n,n)}$ only consists of terms s_γ where $\gamma \in P$ and $l(\gamma) \leq 4$, we have that if $l(\theta) = 5$, then

$$\langle s_{(n,n)} * s_{(n,n)}, s_{\theta/(1)} \rangle = \begin{cases} 1 & \theta_5 = 1, \theta \in P \\ 0 & \text{otherwise.} \end{cases}$$

Summarizing the case $l(\theta) = 5$ we have

$$\langle s_{(n,n,1)} * s_{(n,n,1)}, s_\theta \rangle = \begin{cases} 2d_\theta - 3 + ((\theta \in P)) & \theta_5 = \theta_4 = 1 \\ 2d_\theta - 4 + ((\theta \in P)) & \theta_4 \geq 2, \theta_5 = 1 \\ 1 & \theta_5 = 2 \\ 0 & \text{otherwise.} \end{cases}$$

6.3. Case III: $l(\theta) \leq 4$. Firstly, in this case we know that

$$\langle s_{(n+1,n)} * s_{(n,n,1)}, s_\theta \rangle = d_\theta - 1.$$

Next we consider $\langle s_{(n,n)} * s_{(n,n-1,1)}, s_{\theta/(1)} \rangle$. Recall that, given a partition $\delta \vdash 2n$ and $l(\delta) \leq 4$, we have that

$$\langle s_{(n,n)} * s_{(n,n-1,1)}, s_\delta \rangle = d_\delta - 1.$$

Since $s_{\theta/(1)} = \sum_{\theta^- \prec \theta} s_{\theta^-}$, we obtain

$$\begin{aligned} \langle s_{(n,n)} * s_{(n,n-1,1)}, s_{\theta/(1)} \rangle &= \sum_{\theta^- \prec \theta} (-1 + d_{\theta^-}) \\ &= -d_\theta + \sum_{\theta^- \prec \theta} d_{\theta^-} \\ &= -d_\theta + a_{\theta,2}(d_\theta + 1) + b_{\theta,1}(d_\theta - 1) \\ &\quad + a_{\theta,1}d_\theta + b_{\theta,2}d_\theta. \end{aligned}$$

Since $a_{\theta,1} + a_{\theta,2} + b_{\theta,1} + b_{\theta,2} = d_\theta$, the above equation reduces to

$$\langle s_{(n,n)} * s_{(n,n-1,1)}, s_{\theta/(1)} \rangle = -d_\theta + d_\theta^2 - b_{\theta,1} + a_{\theta,2}.$$

Next we will compute $\langle s_{(n,n)} * s_{(n,n)}, s_{\theta/(1)} \rangle$. If $\theta \vdash 2n + 1$ and $l(\theta) = 4$, then either θ has 3 parts odd and 1 part even, or it has 3 parts even and 1 odd. Since $s_{(n,n)} * s_{(n,n)}$ has terms of the form s_γ where $\gamma \in P$, it is easily seen that if $l(\theta) = 4$, then $\langle s_{(n,n)} * s_{(n,n)}, s_{\theta/(1)} \rangle = 1$.

If $l(\theta) = 3$, then θ has either 3 parts odd, or 2 parts even and 1 odd. In the former case, $s_{\theta/(1)}$ will not have terms of the form s_δ with $\delta \in P$ whereas in the latter, the only term giving a non-zero coefficient is the term s_δ with δ obtained by subtracting 1 from the odd part in θ . Arguments on very similar lines yield that for $l(\theta) \leq 2$, we have $\langle s_{(n,n)} * s_{(n,n)}, s_{\theta/(1)} \rangle = 1$ and thus

$$\langle s_{(n,n)} * s_{(n,n)}, s_{\theta/(1)} \rangle = \begin{cases} 1 & l(\theta) = 4, 2 \text{ or } 1 \\ 1 & l(\theta) = 3 \text{ and } \theta \text{ has exactly 1 odd part} \\ 0 & \text{otherwise.} \end{cases}$$

Since we have covered all cases, we can now give a description for the Kronecker coefficients occurring in $s_{(n,n,1)} * s_{(n,n,1)}$.

Theorem 6.2. *Let $\lambda = (n, n, 1)$ and $\theta \vdash 2n + 1$. Then the Kronecker coefficient $g_{\lambda\lambda}^\theta$ is given as follows.*

$$g_{\lambda\lambda}^\theta = \begin{cases} 1 & l(\theta) = 6, \theta_6 = \theta_5 = 1 \\ 2d_\theta - 3 + ((\theta \in P)) & l(\theta) = 5, \theta_5 = \theta_4 = 1 \\ 2d_\theta - 4 + ((\theta \in P)) & l(\theta) = 5, \theta_4 \geq 2, \theta_5 = 1 \\ 1 & l(\theta) = 5, \theta_5 = 2 \\ (d_\theta - 1)^2 + 1 - b_{\theta,1} + a_{\theta,2} & l(\theta) = 4 \\ (d_\theta - 1)^2 + 1 - b_{\theta,1} + a_{\theta,2} & l(\theta) = 3 \text{ and } \theta \text{ has exactly 1 odd part} \\ (d_\theta - 1)^2 - b_{\theta,1} + a_{\theta,2} & l(\theta) = 3 \text{ and } \theta \text{ has all parts odd} \\ 2 - b_{\theta,1} + a_{\theta,2} & l(\theta) = 2 \\ 1 & l(\theta) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Example 6.3. *Let $\lambda = (8, 8, 1)$. We will compute the coefficients of $s_{(6,5,3,2,1)}$, $s_{(8,6,2,1)}$ and $s_{(7,5,5)}$ in $s_\lambda * s_\lambda$ using Theorem 6.2. For convenience's sake, let $\alpha = (6, 5, 3, 2, 1)$, $\beta = (8, 6, 2, 1)$ and $\gamma = (7, 5, 5)$.*

We will start by computing $g_{\lambda\lambda}^\alpha$. We have that $l(\alpha) = 5$, $\alpha_5 = 1$ and $\alpha_4 \geq 2$. Theorem 6.2 implies that

$$g_{\lambda\lambda}^\alpha = 2d_\alpha - 4 + ((\alpha \in P)).$$

Note also that $\alpha' = (6, 5, 3, 2) \notin P$ and $d_\alpha = 5$. Thus

$$g_{(8,8,1)(8,8,1)}^{(6,5,3,2,1)} = 2 \times 5 - 4 = 6.$$

Next, consider $g_{\lambda\lambda}^\beta$. We have $l(\beta) = d_\beta = 4$. The string σ associated with $(8, 6, 2, 1)$ is 22211. This immediately yields $a_{\beta,2} = 0$ and $b_{\beta,1} = 2$. Theorem 6.2 states that therefore, we have

$$g_{\lambda\lambda}^\beta = (d_\beta - 1)^2 + 1 - b_{\beta,1} + a_{\beta,2}$$

and so

$$g_{(8,8,1)(8,8,1)}^{(8,6,2,1)} = (4 - 1)^2 + 1 - 2 + 0 = 8.$$

Finally, we compute $g_{\lambda\lambda}^\gamma$. Note that $l(\gamma) = 3$, $d_\gamma = 2$ and the string σ associated with $\gamma = (7, 5, 5)$ is 2202. Thus, we have $a_{\gamma,2} = 1$ and $b_{\gamma,1} = 0$. Furthermore, since all parts of γ are odd, by Theorem 6.2, we have that

$$g_{\lambda\lambda}^\gamma = (d_\gamma - 1)^2 - b_{\gamma,1} + a_{\gamma,2}$$

and so

$$g_{(8,8,1)(8,8,1)}^{(7,5,5)} = (2 - 1)^2 - 0 + 1 = 2.$$

7. COMBINATORIAL IMPLICATIONS

A natural question to ask is how many SYTs are there of fixed size n if we impose the constraint that the number of parts of $\lambda \vdash n$ is bounded above by some fixed positive integer k . This means we are interested in the sum

$$\tau_k(n) = \sum_{\substack{\lambda \vdash n \\ l(\lambda) \leq k}} f_\lambda.$$

This is a well-studied question as is evident from [3, 4, 19, 20, 26]. The expressions for $\tau_k(n)$ are unwieldy when k is large. But for relatively small values of k , these expressions are more succinct than what one would expect from the hooklength formula. For example, Regev [26] found the following closed form expressions for $\tau_2(n)$ and $\tau_3(n)$

$$\tau_2(n) = \binom{n}{\lfloor \frac{n}{2} \rfloor}, \quad \tau_3(n) = \sum_{i \geq 0} \frac{1}{i+1} \binom{n}{2i} \binom{2i}{i}, \quad (16)$$

where $\tau_3(n)$ is the Motzkin number M_n . Gessel [19] found an expression for $\tau_4(n)$ while Gouyou-Beauchamps [20] found an expression for both $\tau_4(n)$ and $\tau_5(n)$, namely

$$\tau_4(n) = C_{\lfloor \frac{n+1}{2} \rfloor} C_{\lceil \frac{n+1}{2} \rceil}, \quad \tau_5(n) = 6 \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} C_i \frac{(2i+2)!}{(i+2)!(i+3)!}, \quad (17)$$

where $C_i = \frac{1}{i+1} \binom{2i}{i}$ is the i -th Catalan number.

In this section, we will use our results on Kronecker coefficients to prove a result similar to the aforementioned ones. Given a positive integer n , consider the set L_n defined as follows.

$$L_n = \{\lambda \vdash n : l(\lambda) = 5, \lambda_5 = 1\}$$

In Theorem 7.4, we will give a closed form expression for the sum

$$\sum_{\lambda \in L_n} f_\lambda.$$

Towards this goal, the following proposition is useful for our purposes. The claims therein can be proved easily using the hooklength formula.

Proposition 7.1. *Given $n \geq 2$, we have that*

$$f_{(n,n)} = f_{(n,n-1)} = C_n, \quad (18)$$

$$f_{(n,n-1,1)} = \left(\frac{(n-1)(n+1)}{2n+1} \right) C_{n+1}, \quad (19)$$

$$f_{(n-1,n-1,1)} = \left(\frac{n-1}{2} \right) C_n. \quad (20)$$

Recall that the Kronecker coefficients $g_{\lambda\mu}^\nu$ describe the decomposition of the product of the irreducible characters χ_λ and χ_μ of the symmetric group, i.e., $\chi_\lambda\chi_\mu = \sum_{\nu} g_{\lambda\mu}^\nu\chi_\nu$.

Evaluating the character on the identity element of the symmetric group, denoted by 1, we obtain $\chi_\lambda(1) = f_\lambda$. Therefore, the results obtained about the Kronecker products $s_{(n,n-1,1)} * s_{(n,n)}$ and $s_{(n-1,n-1,1)} * s_{(n,n-1)}$ in Sections 2 and 3 imply the following relations

$$\sum_{\substack{\lambda \vdash 2n \\ l(\lambda) \leq 4}} (d_\lambda - 1)f_\lambda + \sum_{\substack{\lambda \vdash 2n \\ l(\lambda)=5 \\ \lambda_5=1}} f_\lambda = f_{(n,n-1,1)}f_{(n,n)}, \quad (21)$$

$$\sum_{\substack{\lambda \vdash 2n-1 \\ l(\lambda) \leq 4}} (d_\lambda - 1)f_\lambda + \sum_{\substack{\lambda \vdash 2n-1 \\ l(\lambda)=5 \\ \lambda_5=1}} f_\lambda = f_{(n-1,n-1,1)}f_{(n,n-1)}. \quad (22)$$

Now, define $\sigma_k(n)$ as follows.

$$\sigma_k(n) = \sum_{\lambda \vdash n, l(\lambda) \leq k} d_\lambda f_\lambda$$

Next, we will give a simple expression for $\sigma_k(n)$.

Theorem 7.2. *Given positive integers m and k , we have*

$$\sigma_k(m) = \tau_k(m+1) - \tau_{k-1}(m).$$

Proof. By definition we have that

$$\tau_k(m+1) = \sum_{\substack{\lambda \vdash m+1 \\ l(\lambda) \leq k}} f_\lambda.$$

Using [29, Lemma 2.8.2], which says $f_\lambda = \sum_{\mu \prec \lambda} f_\mu$, we have

$$\begin{aligned} \sum_{\substack{\lambda \vdash m+1 \\ l(\lambda) \leq k}} f_\lambda &= \sum_{\substack{\lambda \vdash m+1 \\ l(\lambda) \leq k}} \sum_{\mu \prec \lambda} f_\mu \\ &= \sum_{\substack{\mu \vdash m \\ l(\mu) \leq k}} \sum_{\substack{\lambda \succ \mu \\ l(\lambda) \leq k}} f_\mu \\ &= \sum_{\substack{\mu \vdash m \\ l(\mu) \leq k-1}} (d_\mu + 1)f_\mu + \sum_{\substack{\mu \vdash m \\ l(\mu) = k}} d_\mu f_\mu \\ &= \sum_{\substack{\lambda \vdash m \\ l(\lambda) \leq k}} d_\lambda f_\lambda + \tau_{k-1}(m) \\ &= \sigma_k(m) + \tau_{k-1}(m). \end{aligned}$$

Thus the claim is established. \square

Using Theorem 7.2 in conjunction with known expressions for $\tau_3(n)$ and $\tau_4(n)$ given in (16) and (17) respectively, we get the following corollary.

Corollary 7.3. *Given a positive integer n , we have*

$$\sigma_4(n) = C_{\lfloor \frac{n}{2} \rfloor + 1} C_{\lceil \frac{n}{2} \rceil + 1} - M_n. \quad (23)$$

Now we come to our main enumerative result which makes use of (21) and (22), and is a specific case of counting standard Young tableaux with a fixed height.

Theorem 7.4. *Given a positive integer $k \geq 3$, we have*

$$\sum_{\lambda \in L_k} f_\lambda = \frac{\lfloor \frac{k+1}{2} \rfloor (\lceil \frac{k+1}{2} \rceil + 1)}{k+1} C_{\lfloor \frac{k+1}{2} \rfloor} C_{\lceil \frac{k+1}{2} \rceil} - C_{\lfloor \frac{k}{2} \rfloor + 1} C_{\lceil \frac{k}{2} \rceil + 1} + M_k.$$

Proof. We will treat the cases where k is odd and k is even separately. Firstly assume $k = 2n$ for some integer $n \geq 2$. Then (21) implies

$$\begin{aligned} \sum_{\lambda \in L_{2n}} f_\lambda &= f_{(n, n-1, 1)} f_{(n, n)} - \sum_{\substack{\lambda \vdash 2n \\ l(\lambda) \leq 4}} (d_\lambda - 1) f_\lambda \\ &= f_{(n, n-1, 1)} f_{(n, n)} - \sigma_4(2n) + \tau_4(2n). \end{aligned}$$

Substituting the expressions for $\sigma_4(2n)$, $\tau_4(2n)$, $f_{(n, n)}$, and $f_{(n, n-1, 1)}$ given in equations (23), (17), (18), and (19) into the right hand side of the above formula we obtain

$$\begin{aligned} \sum_{\lambda \in L_{2n}} f_\lambda &= \binom{n^2 - 1}{2n + 1} C_n C_{n+1} - C_{n+1}^2 + M_{2n} + C_n C_{n+1} \\ &= \binom{n(n+2)}{2n+1} C_n C_{n+1} - C_{n+1}^2 + M_{2n}. \end{aligned}$$

Now, assume $k = 2n - 1$ for $n \geq 2$. Then (22) implies

$$\begin{aligned} \sum_{\lambda \in L_{2n-1}} f_\lambda &= f_{(n-1, n-1, 1)} f_{(n, n-1)} - \sum_{\substack{\lambda \vdash 2n-1 \\ l(\lambda) \leq 4}} (d_\lambda - 1) f_\lambda \\ &= f_{(n-1, n-1, 1)} f_{(n, n-1)} - \sigma_4(2n-1) + \tau_4(2n-1). \end{aligned}$$

Substituting the expressions for $\sigma_4(2n-1)$, $\tau_4(2n-1)$, $f_{(n, n-1)}$, and $f_{(n-1, n-1, 1)}$ given in equations (23), (17), (18), and (20) into the right hand side of the above formula we obtain

$$\begin{aligned} \sum_{\lambda \in L_{2n-1}} f_\lambda &= \binom{n-1}{2} C_n^2 - C_n C_{n+1} + M_{2n-1} + C_n^2 \\ &= \binom{n+1}{2} C_n^2 - C_n C_{n+1} + M_{2n-1}. \end{aligned}$$

The claim is now a unified way of rewriting the formulae obtained in the two cases, $k = 2n$ and $k = 2n - 1$. \square

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