General Comments:

- WORDS HAVE MEANING
  Yes, we habitually use words and phrases such as *hence, consider, it follows that, then, since,* etc. to connect our mathematical thoughts. But all of these words mean something, so be sure you are using them properly. Any word or phrase used to show an implication (*hence, therefore, thus, it follows, then,* etc.) should be preceded by a statement that allows us to come to the conclusion given in the statement following the above word or phrase. *Suppose*, *let* and *consider* also have slightly different purposes. I don't know proper grammatical terms and breakdown, but the sentences
  - *Suppose* $\left\{a_n\right\}_{n=1}^{\infty}$ *converges to* A.
  - *Let* $\left\{a_n\right\}_{n=1}^{\infty}$ *be a sequence that converges to* A.
  - *Consider a sequence* $\left\{a_n\right\}_{n=1}^{\infty}$ *that converges to* A.

  all say the same thing, and they are all okay. But,
  - *Let* $\left\{a_n\right\}_{n=1}^{\infty}$ *converges to* A.
  - *Consider* $\left\{a_n\right\}_{n=1}^{\infty}$ *be a sequence that converges to* A.
  - *Suppose a sequence* $\left\{a_n\right\}_{n=1}^{\infty}$ *converge to* A.

  are not proper. Some versions of this are not even sentences. I get what you are saying, but the grammar is not right.
- DEFINITIONS
  You still need to know your definitions. In exercise 24 (and 21) limit points (or points of convergence) are not the exact same thing as accumulation points, so you cannot claim that because something is a limit point it is an accumulation point. The definition of accumulation point states

  > A real number A is an accumulation point of a set of real numbers S iff every neighborhood of A contains infinitely many points of S.

  You NEED INFINITELY MANY POINTS. For example, the sequence $\{x\}_{n=1}^{\infty}$ converges to x for any $x \in \mathbb{R}$. Infinitely many TERMS of this sequence are in any given neighborhood of x, but the range of this sequence does not have an accumulation point because it is finite.
- The first exercise did not ask for justification, so almost everyone got full points. You just had to claim that $\left\{2^i \mid i \in \mathbb{N}\right\}$ is the set of accumulation points. Think about how you would justify it if you did not try. For those of you who did try, are there any holes in your argument? In particular, are there any holes related to the above remark, perhaps?
- A lot of you stated in the second exercise that the sequence had distinct terms. You do not know this. The sequence could be $\left\{1, 1, 2, 2, 3, 3, \ldots\right\}$. The point is that the range of the sequence is infinite – and you did need this to prove the claim.

1. (Chapter 1, exercise 21)
Determine the accumulation points of the set $\left\{2^n + \frac{1}{k} \;\middle|\; n \text{ and } k \text{ are positive integers}\right\}$.

Proof. Denote $S = \left\{2^n + \frac{1}{k} \;\middle|\; n, k \in \mathbb{N}\right\}$, and note

$$S = \left\{2 + \frac{1}{k} \mid k \in \mathbb{N}\right\} \cup \left\{2^2 + \frac{1}{k} \mid k \in \mathbb{N}\right\} \cup \cdots = \bigcup_{i=1}^{\infty} \left\{2^i + \frac{1}{k} \mid k \in \mathbb{N}\right\}.$$

Recall, a real number A is an accumulation point of the set S iff every neighborhood of A contains infinitely many points of S. We claim $\left\{2^i \mid i \in \mathbb{N}\right\}$ is the set of accumulation points, S', of S.

Let P be a neighborhood of 2^i for some $i \in \mathbb{N}$. Then there exists $\epsilon > 0$ such that $(2^i - \epsilon, 2^i + \epsilon) \subset P$. Choose $K \in \mathbb{N}$ so that $\frac{1}{\epsilon} < K$. Note, this K exists by Theorem 0.21. Then for all $k \geq K$, we have $\frac{1}{k} \leq \frac{1}{K} < \epsilon$ and

$$2^i - \epsilon < 2^i - \frac{1}{K} \leq 2^i - \frac{1}{k} < 2^i + \frac{1}{k} \leq 2^i + \frac{1}{K} < 2^i + \epsilon.$$

Hence, $\left(2^i - \epsilon, 2^i + \epsilon\right)$ contains infinitely many points of $\left\{2^i + \frac{1}{k} \;\middle|\; k \in \mathbb{N}\right\}$. Consequently, the arbitrary neighborhood P of 2^i contains infinitely many points of $\bigcup_{i=1}^{\infty} \left\{2^i + \frac{1}{k} \;\middle|\; k \in \mathbb{N}\right\} = \left\{2^n + \frac{1}{k} \;\middle|\; n, k \in \mathbb{N}\right\} = S$, and $\left\{2^i \;\middle|\; i \in \mathbb{N}\right\} = S'$ by definition. □

*Note, I have actually only shown $\left\{2^i \mid i \in \mathbb{N}\right\} \subset S'$. How could you show $S' \subset \left\{2^i \mid i \in \mathbb{N}\right\}$ (that the *only* accumulation points are of the form 2^i)?

2. (Chapter 1, exercise 24)
Suppose $\left\{a_n\right\}_{n=1}^{\infty}$ converges to A and $\left\{a_n \;\middle|\; n \in \mathbb{N}\right\}$ is an infinite set. Show that A is an accumulation point of $\left\{a_n \;\middle|\; n \in \mathbb{N}\right\}$.

Proof. Let P be a neighborhood of A. Then there exists $\epsilon > 0$ such that $(A-\epsilon, A+\epsilon) \subset P$. Since $\left\{a_n\right\}_{n=1}^{\infty}$ converges to A. We know there exists an $N \in \mathbb{N}$, corresponding to this ϵ, such that for all $n \geq N$ we have $\left|a_n - A\right| < \epsilon$. This inequality is equivalent to $A - \epsilon < a_n < A + \epsilon$. Hence, if $n \geq N$, then $a_n \in (A-\epsilon, A+\epsilon)$. Since there in fact do exist infinitely many a_n, this shows that the interval $(A-\epsilon, A+\epsilon)$, and consequently the neighborhood P, contains infinitely many points of S. (We know only $a_1, a_2, \ldots, a_{N-1}$ lie outside of $(A-\epsilon, A+\epsilon)$, so if there were also only finitely many points of S inside as well as outside of $(A-\epsilon, A+\epsilon)$, the set S would be finite, which is a contradiction.) By definition, A is an accumulation point of S as desired. □

* You could also use the lemma on page 35 of the text stating that a sequence $\left\{a_n\right\}_{n=1}^{\infty}$ converges to A iff each neighborhood of A contains all but a finite number of terms of the sequence. Hence, since the range of the sequence is infinite, each neighborhood of A contains infinitely many elements of $\left\{a_n \mid n \in \mathbb{N}\right\}$ and A is an accumulation point of this set by definition.

3. (Chapter 1, exercise 28)
If $\{a_n\}_{n=1}^{\infty}$ converges to a with $a_n \geq 0$ for all n, show $\{\sqrt{a_n}\}_{n=1}^{\infty}$ converges to $\sqrt{a}$. (*Hint:* If $a > 0$, then $\sqrt{a_n} - \sqrt{a} = \frac{a_n - a}{\sqrt{a_n} - \sqrt{a}}$.)

Proof. Note, the constant sequence $\{0\}_{n=1}^{\infty}$ converges to 0, $\{a_n\}_{n=1}^{\infty}$ converges to a, and $0 \leq a_n$ for all n. Hence, by Theorem 1.12, it follows that $0 \leq a$. Thus, we have two cases.

Case I: $a = 0$.

Let $\epsilon > 0$ be given. Since $\{a_n\}_{n=1}^{\infty}$ converges to zero, there exists an $N \in \mathbb{N}$ such that $0 \leq a_n < \epsilon^2$ for all $n \geq N$. By exercise 41 in Chapter 0, we then have that $0 \leq \sqrt{a_n} < \epsilon$ for all $n \geq N$. Hence, $\{\sqrt{a_n}\}_{n=1}^{\infty}$ converges to 0 as desired.

Case II: $a > 0$.

Let $\epsilon > 0$ be given. Since $\{a_n\}_{n=1}^{\infty}$ converges to a, we can choose $N \in \mathbb{N}$ such that $|\, a_n - a \,| < \epsilon\sqrt{a}$ for all $n \geq N$. It follows that if $n \geq N$, then

$$\begin{aligned} |\, \sqrt{a_n} - \sqrt{a} \,| &= \frac{|\, a_n - a \,|}{\sqrt{a_n} + \sqrt{a}} && \text{by the hint and since } a_n \geq 0,\ a > 0 \\ &\leq \frac{|\, a_n - a \,|}{\sqrt{a}} && \text{since } \sqrt{a_n} \geq 0,\ a > 0 \\ &< \frac{\epsilon\sqrt{a}}{\sqrt{a}} && \text{since } a_n \to a \text{ as } n \to \infty \text{ and } n \geq N \\ &= \epsilon. \end{aligned}$$

By definition, $\{\sqrt{a_n}\}_{n=1}^{\infty}$ converges to $\sqrt{a}$ as was to be shown. □