General Comments:

- Some of you in the first problem tried showing that an arbitrary subsequence will converge, but this is not necessarily true. Knowing x is an accumulation point gives you the tools to CONSTRUCT A SUBSEQUENCE (or SHOW THE EXISTENCE OF A SUBSEQUENCE) that converges to x.
- For the second problem, I think a lot of you tried to use the direction of Theorem 1.14 stating, "If each of a sequence's subsequences converges, then the sequence converges." By construction, though, you have only that TWO of the subsequences of $\{z_n\}_{n=1}^{\infty}$ converge, not ALL of the subsequences. What about the subsequences $\{z_{3n}\}_{n=1}^{\infty}$ or $\{z_{n^2}\}_{n=1}^{\infty}$? So you cannot immediately use Theorem 1.14 to conclude that $\{z_n\}_{n=1}^{\infty}$ converges to x_0. This cost almost everyone a lot of points because the improper use of the theorem caused you to not be able to do the exercise at all.
- For the last problem I saw a clever solution utilizing the theorem stating that every real number x is an accumulation point of a set of rational numbers, then using exercise 35 to show the existence of a subsequence of this set of rational numbers that converges to x.

1. (Chapter 1, exercise 35)
Suppose x is an accumulation point of $\{a_n : n \in \mathbb{N}\}$. Show that there is a subsequence of $\{a_n\}_{n=1}^{\infty}$ that converges to x

Proof. Let x be an accumulation point of $\{a_n : n \in \mathbb{N}\}$. Consider the neighborhood of x given by $\left(x - \frac{1}{k}, x + \frac{1}{k}\right)$ where $k \in \mathbb{N}$. By the lemma on page 39, for each $k \in \mathbb{N}$ there is an element $a_{n_k} \neq x$ in $\{a_n : n \in \mathbb{N}\}$ such that $a_{n_k} \in \left(x - \frac{1}{k}, x + \frac{1}{k}\right)$. It follows that the subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ converges to x since given $\epsilon > 0$ we can choose $N > \frac{1}{\epsilon}$ and have

$$|a_{n_k} - x| < \frac{1}{k} \leq \frac{1}{N} < \epsilon$$

for $k \geq N$. □

2. (Chapter 1, exercise 39)
Suppose $\{x_n\}_{n=1}^{\infty}$ converges to x_0 and $\{y_n\}_{n=1}^{\infty}$ converges to x_0. Define a sequence $\{z_n\}_{n=1}^{\infty}$ as follows: $z_{2n} = x_n$ and $z_{2n-1} = y_n$. Prove that $\{z_n\}_{n=1}^{\infty}$ converges to x_0.

Proof. We will first show that $\{z_n\}_{n=1}^{\infty}$ is Cauchy; that is, given $\epsilon > 0$, we will show that there exists $N \in \mathbb{N}$ such that $|z_n - z_m| < \epsilon$ for all $n, m \geq N$. Note, the terms of $\{z_n\}_{n=1}^{\infty}$ consist exactly of terms of $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$. Hence, our options for $|z_n - z_m|$ are as follows

$$|z_n - z_m| = \begin{cases} |x_k - x_\ell| \\ |y_k - y_\ell| \\ |x_k - y_\ell| \end{cases}$$

for some $k, \ell \in \mathbb{N}$. Note, since $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ converge, these sequences are Cauchy (Theorem 1.3). Thus, it suffices to show $|x_k - y_\ell| < \epsilon$.

Let $\epsilon > 0$ be given. Since $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ converge to x_0, there exists $N_1 \in \mathbb{N}$ such that $|x_k - x_0| < \frac{\epsilon}{2}$ for all $k \geq N_1$, and $N_2 \in \mathbb{N}$ such that $|y_\ell - x_0| < \frac{\epsilon}{2}$ for all $\ell \geq N_2$. Choose $N = \max\{N_1, N_2\}$. Then for all $k, \ell \geq N$ we have

$$\begin{aligned} |x_k - y_\ell| &= |x_k - x_0 + x_0 - y_\ell| \\ &\leq |x_k - x_0| + |x_0 - y_\ell| \qquad \text{by the triangle inequality} \\ &= |x_k - x_0| + |y_\ell - x_0| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

It follows by definition that $\{z_n\}_{n=1}^{\infty}$ is Cauchy. By Theorem 1.7, we then know that this sequence converges. Finally, by Theorem 1.14, we have that $\{z_n\}_{n=1}^{\infty}$ must converge to the same limit as all of its subsequences, and hence must converge to x_0 as desired. $\square$

*This could have also been done without any use of information about subsequences, but just with the definition of convergence and proper indexing. It just seems more fumbly and a bit awkward to me:
Let $\epsilon > 0$ be given. Since $z_{2n} = x_n$ and $\{x_n\}_{n=1}^{\infty}$ converges to x_0, we know there is an $N_1 \in \mathbb{N}$ such that $|z_{2n} - x_0| = |x_n - x_0| < \epsilon$ for $n \geq N_1$. Similarly, since $z_{2n-1} = y_n$ and $\{y_n\}_{n=1}^{\infty}$ converges to x_0, we know there is an $N_2 \in \mathbb{N}$ such that $|z_{2n-1} - x_0| < \epsilon$ for $n \geq N_2$. Thus, set $N = \max\{2N_1, 2N_2 - 1\}$. Then for all $n \geq N$, we have $n \geq 2N_1$ AND $n \geq 2N_2 - 1$ which implies

$$|z_n - x_0| \leq |x_{N_1} - x_0| < \epsilon \quad \text{because } x_{N_1} = z_{2N_1} \text{ and we are beyond } z_{2N_1} \text{ in the sequence}$$

AND

$$|z_n - x_0| \leq |y_{N_2} - x_0| < \epsilon \quad \text{because } y_{N_2} = z_{2N_2-1} \text{ and we are beyond } z_{2N_2-1} \text{ in the sequence.}$$

Since the terms of $\{z_n\}_{n=1}^{\infty}$ comprise exactly of the terms of $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ we are done.

*There is a subtle difference here from some of the arguments submitted in homework showing

$$|z_{2n} - x_0| < \epsilon \quad \text{and} \quad |Z_{2n-1} - x_0| < \epsilon$$

when $n \geq \max\{N_1, N_2\}$. Just taking the max of the $N's$ that work for the x_n and y_n sequences does not guarantee that you will be far out enough in the sequence of z_n to be within an epsilon neighborhood of x_0.

3. (Chapter 1, exercise 45)
Show that if x is any real number, there is a sequence of rational numbers converging to x.

Proof. Let $x \in \mathbb{R}$. By Theorem 0.22, we know there exists $a_n \in \mathbb{Q}$ such that $x - \frac{1}{n} < a_n < x + \frac{1}{n}$, where $n \in \mathbb{N}$. Note, this is equivalent to $|a_n - x| < \frac{1}{n}$. Thus, given $\epsilon > 0$, choose $N > \frac{1}{\epsilon}$. Then for all $n \geq N$ we have

$$|a_n - x| < \frac{1}{n} \leq \frac{1}{N} < \epsilon,$$

showing that the sequence $\{a_n\}_{n=1}^{\infty}$ of rational numbers converges to x as was to be shown. $\square$

*Note, this is not a "well-defined" sequence in the sense that you do not know what rational number a_n I am choosing in the interval $\left(x - \frac{1}{n}, x + \frac{1}{n}\right)$, but that doesn't matter because whatever the choice is, I am still getting closer and closer to x via rational numbers. It is actually well-defined as a function from $\mathbb{N}$ to $\mathbb{Q} \subset \mathbb{R}$ because for each $n \in \mathbb{N}$ I am only choosing one $a_n \in \mathbb{Q}$.