MATH 308 HOMEWORK 6, DUE MARCH 2

Instructions

- Write your name, Math 308, and the homework number.
  - Write down the names of the people you worked with.
  - Write down any resources you used other than ones that most of your classmates would be familiar with, such as the textbook or Wikipedia.
  - Write down the number of hours it took you to complete this assignment.
- Explain how to solve each problem.
  - Write your solutions as if they are to be the published solutions to this problem set. Imagine that you are explaining to your peers how to approach the problem. Your imaginary audience of peers should come away convinced that your solution is correct and capable of solving similar problems on their own.
  - Answers alone won't get any credit, especially since you can often find them in the back of the book.
- Hand in your homework in class.

Problems

(1) Do the following problem that didn't quite make it onto the midterm. (It's the right difficulty, but the computations are too long to do accurately in 10 minutes.)

A tub of ice cream has a radius of 10 cm and a height of 20 cm. Using the center of the base of the tub as the origin, the temperature T inside the tub is

$$T = \tfrac{1}{10}(x^2 + y^2 - 100)\sin\left(\tfrac{\pi}{20}z\right),$$

where T is in degrees Celsius and x, y, and z are in centimeters.

(a) Consider the center of the tub, that is, the point $(x, y, z) = (0, 0, 10)$. Consider a particle that passes through the center of the tub with velocity $3\mathbf{i} - 3\mathbf{j} - 3\mathbf{k}$, with units in centimeters per second. At what rate is the temperature changing when the particle passes through the center of the tub? Specify the units of your answer.

Solution: From class, we know that the rate of temperature change has the formula $\nabla T \cdot \mathbf{v}$, where $\mathbf{v}$ is the velocity. Computing ∇T, we find that

$$\nabla T = \tfrac{1}{10}(2x)\sin\left(\tfrac{\pi}{20}z\right)\mathbf{i} + \tfrac{1}{10}(2y)\sin\left(\tfrac{\pi}{20}z\right)\mathbf{j} + \tfrac{1}{10}(x^2 + y^2 - 100)\tfrac{\pi}{20}\cos\left(\tfrac{\pi}{20}z\right)\mathbf{k}.$$

Plugging in $(x, y, z) = (0, 0, 10)$, we find that $\nabla T = \mathbf{0}$ at the center of the tub, using the fact that $\cos\left(\frac{\pi}{20} \cdot 10\right) = \cos\frac{\pi}{2} = 0$.
The gradient being zero means that the directional derivative in any direction is zero, and in particular, $\nabla T \cdot \mathbf{v} = \boxed{0}$. The actual value of the velocity $\mathbf{v}$ is irrelevant. Zero does not need units, but, if provided, the appropriate units here are degrees Celsius per second.

(b) The formula for the gradient in cylindrical coordinates is

$$\nabla f = \mathbf{e}_r \frac{\partial f}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta} + \mathbf{e}_z \frac{\partial f}{\partial z}.$$

Use this formula to write ∇T in cylindrical coordinates.

(We are no longer focusing on just the center of the tub, so your answer should compute ∇T at an arbitrary point in terms of the cylindrical coordinate variables (r, θ, z).)

Solution: The first step is to rewrite T in cylindrical coordinates using $x^2 + y^2 = r^2$. We write that

$$T = \tfrac{1}{10}(r^2 - 100)\sin\left(\tfrac{\pi}{20}z\right).$$

Next, we apply the formula to find that

$$\begin{aligned}\nabla T &= \mathbf{e}_r \tfrac{1}{10}(2r)\sin\left(\tfrac{\pi}{20}z\right) + 0 + \mathbf{e}_z \tfrac{1}{10}(r^2 - 100)\tfrac{\pi}{20}\cos\left(\tfrac{\pi}{20}z\right) \\ &= \boxed{\tfrac{1}{5}r\sin\left(\tfrac{\pi}{20}z\right)\mathbf{e}_r + \tfrac{\pi}{200}(r^2 - 100)\cos\left(\tfrac{\pi}{20}z\right)\mathbf{e}_z.}\end{aligned}$$

As a sanity check, we can check that when $r = 0$ and $z = 10$, we get zero, matching our rectangular coordinate computation in the previous part.

(c) Compute ΔT, the Laplacian of the temperature.

Solution: The Laplacian has a fairly complicated formula in cylindrical coordinates, so we go back to rectangular coordinates. We already computed the first derivatives of T above, so, computing the second derivatives, we find that

$$\begin{aligned}\Delta T &= \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \\ &= \tfrac{1}{5}\sin\left(\tfrac{\pi}{20}z\right) + \tfrac{1}{5}\sin\left(\tfrac{\pi}{20}z\right) - \tfrac{1}{10}(x^2 + y^2 - 100)\tfrac{\pi^2}{400}\sin\left(\tfrac{\pi}{20}z\right) \\ &= \boxed{\left(\tfrac{2}{5} - \tfrac{\pi^2}{4000}(x^2 + y^2 - 100)\right)\sin\left(\tfrac{\pi}{20}z\right).}\end{aligned}$$

(d) Describe which parts of the tub, if any, are getting warmer. Describe which parts of the tub, if any, are getting colder.

Solution: Our knowledge of the heat equation tells us that the parts of the tub where the Laplacian is positive get warmer and parts of the tub where the Laplacian is negative are getting colder.

Considering the signs of the various factors, we see that for $0 \le z \le 20$, we have $0 \le \frac{\pi}{20}z \le \pi$, so $\sin\left(\frac{\pi}{20}z\right)$ is positive everywhere in the tub except at its top and bottom boundaries, where it is zero. Meanwhile, since the tub has radius 10, we know that $x^2 + y^2 \le 10^2$, so $x^2 + y^2 - 100 \le 0$. Thus, the $\left(\frac{2}{5} - \frac{\pi^2}{4000}(x^2 + y^2 - 100)\right)$ factor is strictly positive everywhere.

We conclude that $\Delta T > 0$ in the interior of the tub and on the side boundary of the tub, and $\Delta T = 0$ on the top and bottom boundarues of the tub. Thus, the entire tub except for the top and bottom boundary is getting warmer. No part of the tub is getting colder.

(2) In chapter 6, section 9, do problem 11.

(3) In chapter 6, section 10, do problems 1, 5, and 7.