# Math 308 Midterm 1

February 21, 2018

Name: ______________________________

- Show your work. If you solve a problem with anything other than a straightforward computation, write one complete sentence explaining what you're doing.
  - For example, if you're computing a cross product using the standard method, just show your computation.
  - But if, for example, you find that a line integral is zero without actually computing the line integral, you need to write one complete sentence convincing an imaginary peer that that's true.
- Use the back of the previous page for scratchwork. By default, I won't grade the scratchwork, so you can write wrong things there without penalty.
- If you run out of space on the printed page and need more space, then use the back of the previous page, but make sure to:
  - Make a note on the printed page that your work continues on the back of the previous page.
  - On the back of the previous page, put a box around the work that you want graded.
- There are five questions, worth between 10 and 40 points each.
  - Problems worth more points are likely to take more time, but won't necessarily be more difficult.
  - The problems are ordered by topic.

1. Let $\mathbf{A} = -3\mathbf{i} - 2\mathbf{k}$. Let $\mathbf{B} = -3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$.

   (a) (10 points) Compute $\mathbf{A} \cdot \mathbf{B}$ and $\mathbf{A} \times \mathbf{B}$.

   **Solution:** We compute that

   $$\mathbf{A} \cdot \mathbf{B} = (-3)(-3) + (0)(1) + (-2)(2) = 9 - 4 = \boxed{5}.$$

   The cross product is

   $$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 0 & -2 \\ -3 & 1 & 2 \end{vmatrix} = -(-2)(1)\mathbf{i} + (-2)(-3)\mathbf{j} - (-3)(2)\mathbf{j} + (-3)(1)\mathbf{k}$$
   $$= \boxed{2\mathbf{i} + 12\mathbf{j} - 3\mathbf{k}}.$$

   (b) (10 points) Let $\mathbf{A}'$ and $\mathbf{B}'$ be the reflections of $\mathbf{A}$ and $\mathbf{B}$ across the $yz$-plane. Compute $\mathbf{A}' \cdot \mathbf{B}'$ and $\mathbf{A}' \times \mathbf{B}'$.

   Don't forget to justify your answer with one complete sentence if needed.

   **Solution:** Reflecting two vectors does not change their dot product, so $\mathbf{A}' \cdot \mathbf{B}' = A \cdot B = \boxed{5}$.

   A reflection across the $yz$-plane flips the $x$-axis, so the map is $\mathbf{i} \mapsto -\mathbf{i}$, $\mathbf{j} \mapsto \mathbf{j}$, $\mathbf{k} \mapsto \mathbf{k}$. We could use this fact to compute $\mathbf{A}'$ and $\mathbf{B}'$ directly and then compute $\mathbf{A}' \times \mathbf{B}'$, but we can solve the problem faster is we remember from class that if $\mathbf{A} \times \mathbf{B} = \mathbf{C}$, then $\mathbf{A}' \times \mathbf{B}' = -\mathbf{C}'$, where $\mathbf{A}'$, $\mathbf{B}'$, and $\mathbf{C}'$ are the reflections of $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$. The reflection of $2\mathbf{i} + 12\mathbf{j} - 3\mathbf{k}$ is just $-2\mathbf{i} + 12\mathbf{j} - 3\mathbf{k}$. Thus, because $\mathbf{A} \times \mathbf{B} = 2\mathbf{i} + 12\mathbf{j} - 3\mathbf{k}$, we know that

   $$\mathbf{A}' \times \mathbf{B}' = -(-2\mathbf{i} + 12\mathbf{j} - 3\mathbf{k}) = \boxed{2\mathbf{i} - 12\mathbf{j} + 3\mathbf{k}}.$$

2. (15 points) The comet 67P completes a full rotation every 12.4 hours. For a safe landing, the velocity vector of the lander should be close to the velocity vector of the point on the comet that it's trying to land on as the comet rotates. In this problem, you'll determine that velocity vector.

   For the purposes of the problem, approximate this rotation speed as 0.5 radians/hour, and set up the coordinate system so that the axis of rotation is the $z$-axis and the rotation direction is counterclockwise in the $xy$-plane. For the purposes of this problem, the landing site will be at $2\mathbf{i} - 4\mathbf{j} + \mathbf{k}$, with units in kilometers.

   What is the velocity vector of the landing site due to the comet's rotation? Specify the units of your answer.

**Solution:** From the textbook or from class, we have the formula that $\mathbf{v} = \omega \times \mathbf{r}$, where $\mathbf{v}$ is the velocity of the point $\mathbf{r}$ located on an object whose angular velocity is represented by the vector $\omega$.

In this case, the problem is set up so that $\omega = 0.5\mathbf{k}$, with units in radians per hour, and we are given that $\mathbf{r} = 2\mathbf{i} - 4\mathbf{j} + \mathbf{k}$. We then compute that

$$\mathbf{v} = \omega \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 0.5 \\ 2 & -4 & 1 \end{vmatrix} = \mathbf{j}(0.5)(2) - \mathbf{i}(0.5)(-4) = \boxed{2\mathbf{i} + \mathbf{j}},$$

where the units are kilometers per hour.

3. (10 points) Let $\mathbf{A} = 3\mathbf{i} + 3\mathbf{j}$, let $\mathbf{B} = -2\mathbf{i} + \mathbf{k}$, and let $\mathbf{C} = -3\mathbf{j} - 3\mathbf{k}$. Compute $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$. Don't forget to justify your answer with one complete sentence if needed.

**Solution:** We could compute $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$, but, ideally, we remember the triple vector product formula, which makes this computation much less painful. We find that

$$\begin{aligned} \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \\ &= (0 - 9 + 0)\mathbf{B} - (-6 + 0 + 0)\mathbf{C} \\ &= -9(-2\mathbf{i} + \mathbf{k}) + 6(-3\mathbf{j} - 3\mathbf{k}) \\ &= \boxed{18\mathbf{i} - 18\mathbf{j} - 27\mathbf{k}}. \end{aligned}$$

4. Let $\mathbf{V}$ be the vector field $-xy\,\mathbf{i} + y^2\,\mathbf{j} - xy\,\mathbf{k}$.

   (a) (10 points) Compute $\operatorname{div}\mathbf{V}$.

   **Solution:** Using the formula for divergence, we find that

$$\operatorname{div}\mathbf{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = -y + 2y + 0 = \boxed{y}.$$

   (b) (5 points) Interpreting $\mathbf{V}$ as the velocity vector field of a gas, describe the region, if any, where the gas is expanding. Describe the region, if any, where the gas is contracting.

   **Solution:** When the divergence is positive, there is expansion, and when the divergence is negative, there is contraction. Thus, in this case, the gas is expanding when $y > 0$, and the gas is contracting when $y < 0$.

5. Consider the two force fields

$$F_1 = y\mathbf{i} + x\mathbf{j} + \mathbf{k},$$
$$F_2 = \mathbf{i} + y\mathbf{j} + x\mathbf{k}.$$

(a) (10 points) Exactly one of the two force fields is conservative. Which one is it?

**Solution:** These vector fields are defined on the entire plane, so we can figure out whether they are conservative or not by computing whether $\operatorname{curl}\mathbf{F} = 0$ or not. We compute

$$\operatorname{curl}\mathbf{F}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & 1 \end{vmatrix} = \left(\frac{\partial}{\partial x}x - \frac{\partial}{\partial y}y\right)\mathbf{k} = \mathbf{0},$$

making use of the fact that many of the terms are zero. Thus, $\boxed{\mathbf{F}_1}$ is conservative. For completeness, we compute

$$\operatorname{curl}\mathbf{F}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & y & x \end{vmatrix} = -\mathbf{j}\frac{\partial}{\partial x}x = -\mathbf{j} \neq \mathbf{0},$$

again making use of the fact that many of the terms are zero.

(b) (15 points) For the conservative force field, compute a scalar potential. That is, find a scalar field $\phi$ such that $\mathbf{F} = -\nabla\phi$.
Note the negative sign. This $\phi$ is the same as the $U$ we talked about in class, and the negative of the $W$ we talked about.

**Solution:** To compute $W$, we first need to set a reference value. For simplicity, we set $W(0,0,0) = 0$. Then, to compute $W(X,Y,Z)$, we need to compute the line integral from $(0,0,0)$ to $(X,Y,Z)$, along any path. We compute that

$$\mathbf{F}_1 \cdot d\mathbf{r} = y\,dx + x\,dy + dz.$$

Using the straight line path from $(0,0,0)$ to $(X,Y,Z)$, we parametrize it with $x = Xt$, $y = Yt$, and $z = Zt$, for $0 \leq t \leq 1$. Then we have that $dx = X\,dt$, $dy = Y\,dt$, $dz = Z\,dt$. Thus, along this linear path,

$$\mathbf{F}_1 \cdot d\mathbf{r} = (Yt)(X\,dt) + (Xt)(Y\,dt) + Z\,dt = (2XYt + Z)\,dt.$$

Integrating, we find that

$$W(X,Y,Z) = \int_0^1 (2XYt + Z)\,dt = XY + Z.$$

Thus, $W(x,y,z) = xy + z$.
To check, we compute that $\nabla W$ is indeed $y\mathbf{i} + x\mathbf{j} + \mathbf{k}$. Our answer is $\phi = -W = \boxed{-(xy+z)}$.

(c) (15 points) For the other force field, compute

$$\oint_\gamma \mathbf{F} \cdot d\mathbf{r},$$

where $\gamma$ is the circle of radius 2 in the $xy$-plane centered at $(x, y, z) = (2, 2, 0)$, oriented counterclockwise.

**Solution:** Here,

$$\mathbf{F}_2 \cdot d\mathbf{r} = dx + y\,dy + x\,dz.$$

We can parametrize the circle with $x = 2 + 2\cos t$, $y = 2 + 2\sin t$, and $z = 0$ for $0 \le t \le 2\pi$. Thus, $dx = -2\sin t\,dt$, $dy = 2\cos t\,dt$, and $dz = 0$. Thus, along the circle,

$$\begin{aligned}\mathbf{F}_2 \cdot d\mathbf{r} &= -2\sin t\,dt + (2 + 2\sin t)(2\cos t\,dt) + (2 + 2\cos t)(0) \\ &= (-2\sin t + 4\cos t + 4\sin t\cos t)\,dt.\end{aligned}$$

We can see that $\int_0^{2\pi}(-2\sin t + 4\cos t + 4\sin t\cos t)\,dt = \boxed{0}$ by noting that the integral of both cosine and sine is zero over a full period and that $4\sin t\cos t = 2\sin 2t$.

As we'll learn later, that's not surprising. We computed that $\operatorname{curl} \mathbf{F}_2$ is in the $\mathbf{j}$ direction, so that represents rotation in the $xz$-plane. Since our circle is instead in the $xy$-plane, it cannot detect that rotation.

| Question | Points | Score |
|---|---|---|
| 1 | 20 | |
| 2 | 15 | |
| 3 | 10 | |
| 4 | 15 | |
| 5 | 40 | |
| Total: | 100 | |