HOMEWORK 11 SOLUTIONS

MATH 5052, SPRING 2019

Exercise 1 (Folland, Exercise 6.2)**.** Prove Theorem 6.8.

a. If f and g are measurable functions on X, then $\|fg\|_1 \le \|f\|_1 \|g\|_\infty$. If $f \in L^1$ and $g \in L^\infty$, $\|fg\|_1 = \|f\|_1 \|g\|_\infty$ iff $|g(x)| = \|g\|_\infty$ a.e. on the set where $f(x) \neq 0$.

Proof. Because the infimum in the definition of $\|f\|_\infty$ is attained, we have that $|g| \le \|g\|_\infty$ a.e., so $|f|\,|g| \le |f|\,\|g\|_\infty$ a.e., and so

$$\|fg\|_1 = \int |f|\,|g| \le \int |f|\,\|g\|_\infty = \left(\int |f|\right) \|g\|_\infty = \|f\|_1 \|g\|_\infty .$$

To determine when we have equality, observe that $|f|\,\|g\|_\infty - |f|\,|g| \ge 0$, so

$$\int (|f|\,\|g\|_\infty - |f|\,|g|) = 0$$

if and only if

$$0 = |f|\,\|g\|_\infty - |f|\,|g| = |f|\,(\|g\|_\infty - |g|) \text{ a.e..}$$

This means that $f = 0$ or $|g| = \|g\|_\infty$ a.e.. □

b. $\|\cdot\|_\infty$ is a norm on L^∞.

Proof. To check nondegeneracy, assume that $\|f\|_\infty = 0$. Then $|f| \le 0$ a.e., i.e. $f = 0$ a.e.

Checking scaling is clear when $\lambda = 0$. When $\lambda \neq 0$, observe that

$$|\lambda f| = |\lambda|\,|f| \le |\lambda|\,\|f\|_\infty \text{ a.e.,}$$

so $\|\lambda f\|_\infty \le |\lambda|\,\|f\|_\infty$. Applying this result to λ^{-1} and λf, we find that $\|f\|_\infty \le \left|\lambda^{-1}\right| \|\lambda f\|_\infty$. Multiplying by $|\lambda|$ we find $|\lambda|\,\|f\|_\infty \le \|\lambda f\|_\infty$.

To check the triangle inequality, observe that

$$|f+g| \le |f| + |g| \le \|f\|_\infty + \|g\|_\infty \text{ a.e.,}$$

so $\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty$. □

c. $\|f_n - f\|_\infty \to 0$ iff there exists $E \in \mathcal{M}$ such that $\mu(E^c) = 0$ and $f_n \to f$ uniformly on E.

Proof. Assume that $\|f_n - f\|_\infty \to 0$. We know that $|f_n - f| \le \|f_n - f\|_\infty$ a.e.. Let F_n be the measure zero set where this fails, let $F = \bigcup F_n$, and let $E = F^c$. Then $E^c = F$ has measure zero, as desired. Moreover, $|f_n(x) - f(x)| \le \|f_n - f\|_\infty$ for all $x \in E$, so $\|f_n - f\|_{u(E)} \le \|f_n - f\|_{L^\infty(X)} \to 0$, where $\|\cdot\|_{u(E)}$ denotes the uniform norm on E. That is, f_n converges to f uniformly on E.

Conversely, assume $f_n \to f$ uniformly on E and E^c has measure zero. We have that $|f_n(x) - f(x)| \le \|f_n - f\|_{u(E)}$ for all $x \in E$, which means that $|f_n - f| \le \|f_n - f\|_{u(E)}$ a.e., which means that $\|f_n - f\|_{L^\infty(X)} \le \|f_n - f\|_{u(E)}$. By assumption, $\|f_n - f\|_{u(E)} \to 0$, so $\|f_n - f\|_{L^\infty(X)} \to 0$ as well. □

d. L^∞ is a Banach space.

Proof. We make use of part (c) as well as the fact that the space of bounded functions is complete with respect to the uniform norm.

Assume that $\{f_n\}$ is cauchy with respect to the $L^\infty(X)$ norm. We have that $|f_n - f_m| \le \|f_n - f_m\|_\infty$ a.e.. Let $F_{n,m}$ be the measure zero set where this fails, let $F = \bigcup F_{n,m}$, so F has measure zero, and let $E = F^c$. We thus have that $|f_n(x) - f_m(x)| \le \|f_n - f_m\|_\infty$ for all $x \in E$. In other words, $\|f_n - f_m\|_{u(E)} \le \|f_n - f_m\|_{L^\infty(X)}$, which goes to zero as $n, m \to \infty$ by assumption. We conclude that $\{f_n\}$ is Cauchy with respect to the uniform norm on E, so it has a uniform limit f defined on E. Extend the definition of f to all of X arbitrarily. Since $f_n \to f$ uniformly on E and E^c has measure zero, we conclude by part (c) that $f_n \to f$ in $L^\infty(X)$. Thus every sequence that is Cauchy with respect to $L^\infty(X)$ converges with respect to $L^\infty(X)$, so $L^\infty(X)$ is a Banach space. □

e. The simple functions are dense in L^∞.

Proof. We motivate ourselves with the proof of this fact for L^1, which is Theorem 2.26. We note that Theorem 2.26 relies on Theorem 2.10, which is what we use now.

Let $f \in L^\infty$. Since f is measurable, there exists a sequence $\{\phi_n\}$ of simple functions such that $0 \le |\phi_1| \le |\phi_2| \le \cdots \le |f|$, $\phi_n \to f$ pointwise, and $\phi_n \to f$ uniformly on any set on which f is bounded.

We have that $|f| \le \|f\|_\infty$ a.e.. Let E be the set where this holds, so E^c has measure zero. This statement means that f is bounded on E, so $\phi_n \to f$ uniformly on E. By part (c), we conclude that $\phi_n \to f$ in $L^\infty(X)$. □

Exercise 2 (Folland, Exercise 6.7)**.** If $f \in L^p \cap L^\infty$ for some $p < \infty$, so that $f \in L^q$ for all $q > p$, then $\|f\|_\infty = \lim_{q\to\infty} \|f\|_q$.

Proof. By Proposition 6.10, we have $\|f\|_q \le \|f\|_p^{p/q} \|f\|_\infty^{1-p/q}$, which implies $\lim_{q\to\infty} \|f\|_q \le \|f\|_\infty$. Conversely, given $0 < a < \|f\|_\infty$, let $E_a = \big\{x : |f(x)| > a\big\}$, so that $\mu(E_a) > 0$. Then

$$\lim_{q\to\infty} \|f\|_q \ge \lim_{q\to\infty} a\mu(E_a)^{1/q} = a,$$

so taking $a \to \|f\|_\infty$ implies $\lim_{q\to\infty} \|f\|_q \ge \|f\|_\infty$. □

Exercise 3 (Folland, Exercise 6.9)**.** Suppose $1 \leq p < \infty$. If $\|f_n - f\|_p \to 0$, then $f_n \to f$ in measure, and hence some subsequence converges to f a.e. On the other hand, if $f_n \to f$ in measure and $|f_n| \leq g \in L^p$ for all n, then $\|f_n - f\|_p \to 0$.

Proof. Let $E_{n,\epsilon} = \{x : |f_n(x) - f(x)| \geq \epsilon\}$ (as in the proof of Proposition 2.29). Then

$$\int |f_n - f|^p \geq \int_{E_{n,\epsilon}} |f_n - f|^p \geq \epsilon^p \mu(E_{\epsilon,n}),$$

and hence $\mu(E_{\epsilon,n}) \leq \epsilon^{-p} \|f_n - f\|_p^p \to 0$. Conversely, if $f_n \to f$ in measure and $|f_n| \leq g \in L^p$, then the same is true of any subsequence $\{f_{n_j}\}$. Hence, we can take a further subsequence $\{f_{n_{j_k}}\}$ which converges to f a.e., so the dominated convergence theorem implies that $\int |f_{n_{j_k}} - f|^p \to 0$, and since the original subsequence was arbitrary, $\|f_n - f\|_p \to 0$.

(The above argument essentially repeats the earlier results relating convergence in measure to L^1 convergence. Alternatively, we could just observe that $\|f_n - f\|_p \to 0$ iff $\big\| |f_n - f|^p \big\|_1 \to 0$, and apply the earlier results directly.) □

Exercise 4 (Folland, Exercise 6.12)**.** If $p \neq 2$, the L^p norm does not arise from an inner product on L^p, except in trivial cases when $\dim(L^p) \leq 1$.

Proof. If $\dim(L^p) \geq 2$, then we can pick two simple functions $a\chi_E$ and $b\chi_F$ with $E \cap F = \varnothing$ and $a\mu(E) = b\mu(F) = 1$. It follows that

$$\|a\chi_E \pm b\chi_F\|_p = \big(a\mu(E) + b\mu(F)\big)^{1/p} = 2^{1/p},$$

so

$$\frac{\|a\chi_E + b\chi_F\|_p^2 + \|a\chi_E - b\chi_F\|_p^2}{2\big(\|a\chi_E\|_p^2 + \|b\chi_F\|_p^2\big)} = \frac{2(2^{2/p})}{4} = 2^{2/p-1},$$

which equals 1 iff $p = 2$. □