HOMEWORK 3 SOLUTIONS

MATH 5052, SPRING 2019

Exercise 1 (Folland, Exercise 5.17)**.** A linear functional f on a normed vector space $\mathcal{X}$ is bounded iff $f^{-1}(\{0\})$ is closed.

Proof. ($\Rightarrow$) If f is bounded, then it is continuous, so the preimage of the closed set $\{0\}$ is also closed.

($\Leftarrow$) Suppose that $f^{-1}(\{0\})$ is closed. If $f^{-1}(\{0\}) = \mathcal{X}$, then $f = 0$, which is clearly bounded. Otherwise, $\mathcal{M} = f^{-1}(\{0\})$ is a proper closed subspace, so by Exercise 12b, there exists $x \in \mathcal{X}$ such that $\|x\| = 1$ and $\|x + \mathcal{M}\| > \frac{1}{2}$. Therefore, if $\|y\| \leq \frac{1}{2}$, we have $x + y \notin \mathcal{M}$. Since f is linear, the image $\{f(y) : \|y\| \leq \frac{1}{2}\}$ is connected and symmetric about zero. Hence, it is either bounded, in which case we're done, or equal to all of $\mathbb{R}$, in which case $f(x + y) = f(x) + f(y) = 0$ for some $\|y\| \leq \frac{1}{2}$, contradicting $x + y \notin \mathcal{M}$. □

Exercise 2 (Folland, Exercise 5.19)**.** Let $\mathcal{X}$ be an infinite-dimensional normed vector space.

a. There is a sequence $\{x_j\}$ in $\mathcal{X}$ such that $\|x_j\| = 1$ for all j and $\|x_j - x_k\| \geq \frac{1}{2}$ for $j \neq k$.

Proof. Start by choosing some $x_1 \in \mathcal{X}$ such that $\|x_1\| = 1$. Inductively, given $x_1, \ldots, x_n$ satisfying this condition, consider the subspace $\mathcal{M} = \operatorname{span}\{x_1, \ldots, x_n\}$, which is finite-dimensional and hence closed. Therefore, Exercise 12b with $\epsilon = \frac{1}{2}$ implies that there exists $x_{n+1} \in \mathcal{X} \setminus \mathcal{M}$ such that $\|x_{n+1}\| = 1$ and $\|x_{n+1} + \mathcal{M}\| \geq \frac{1}{2}$. □

b. $\mathcal{X}$ is not locally compact.

Proof. The disjoint collection of balls $\{B(\frac{1}{2}, x_j)\}_{j=1}^{\infty}$ covers $\{x_j\}_{j=1}^{\infty}$, but has no finite subcover. Hence, the closed unit ball is not compact, so by the continuity of translation and scalar multiplication, neither is any neighborhood of a point $x \in \mathcal{X}$. □