Problem A1

Find the smallest positive integer $j$ such that for every polynomial $p(x)$ with integer coefficients and for every integer $k$, the integer

\[
p^{(j)}(k) = \frac{d^j}{dx^j} p(x)_{x=k}
\]

(the $j$-th derivative of $p(x)$ at $k$) is divisible by 2016.
Problem A2

Given a positive integer $n$, let $M(n)$ be the largest integer $m$ such that

$$\binom{m}{n-1} > \binom{m-1}{n}.$$ 

Evaluate

$$\lim_{n \to \infty} \frac{M(n)}{n}.$$
Problem A3

Suppose that $f$ is a function from $\mathbb{R}$ to $\mathbb{R}$ such that

$$f(x) + f\left(1 - \frac{1}{x}\right) = \arctan x$$

for all real $x \neq 0$. (As usual, $y = \arctan x$ means $-\pi/2 < y < \pi/2$ and $\tan y = x$.)

Find

$$\int_0^1 f(x) \, dx.$$
Problem A4

Consider a \((2m - 1) \times (2n - 1)\) rectangular region, where \(m\) and \(n\) are integers such that \(m, n \geq 4\). This region is to be tiled using tiles of the two types shown:

![Tiles](image)

(The dotted lines divide the tiles into \(1 \times 1\) squares.) The tiles may be rotated and reflected, as long as their sides are parallel to the sides of the rectangular region. They must all fit within the region, and they must cover it completely without overlapping.

What is the minimum number of tiles required to tile the region?
Problem A5

Suppose that $G$ is a finite group generated by the two elements $g$ and $h$, where the order of $g$ is odd. Show that every element of $G$ can be written in the form

$$g^{m_1}h^{n_1}g^{m_2}h^{n_2} \cdots g^{m_r}h^{n_r},$$

with $1 \leq r \leq |G|$ and $m_1, n_1, m_2, n_2, \ldots, m_r, n_r \in \{1, -1\}$. (Here $|G|$ is the number of elements of $G$.)
Problem A6

Find the smallest constant $C$ such that for every real polynomial $P(x)$ of degree 3 that has a root in the interval $[0,1]$, 

$$\int_0^1 |P(x)| \, dx \leq C \max_{x \in [0,1]} |P(x)|.$$
Problem B1

Let $x_0, x_1, x_2, \ldots$ be the sequence such that $x_0 = 1$ and for $n \geq 0$,

$$x_{n+1} = \ln \left( e^{x_n} - x_n \right)$$

(as usual, the function $\ln$ is the natural logarithm). Show that the infinite series

$$x_0 + x_1 + x_2 + \cdots$$

converges and find its sum.
Problem B2

Define a positive integer \( n \) to be *squirish* if either \( n \) is itself a perfect square or the distance from \( n \) to the nearest perfect square is a perfect square. For example, 2016 is squirish, because the nearest perfect square to 2016 is \( 45^2 = 2025 \) and \( 2025 - 2016 = 9 \) is a perfect square. (Of the positive integers between 1 and 10, only 6 and 7 are not squirish.)

For a positive integer \( N \), let \( S(N) \) be the number of squirish integers between 1 and \( N \), inclusive. Find positive constants \( \alpha \) and \( \beta \) such that

\[
\lim_{N \to \infty} \frac{S(N)}{N^\alpha} = \beta,
\]

or show that no such constants exist.
Problem B3

Suppose that \( S \) is a finite set of points in the plane such that the area of triangle \( \triangle ABC \) is at most 1 whenever \( A, B, \) and \( C \) are in \( S \). Show that there exists a triangle of area 4 that (together with its interior) covers the set \( S \).
Problem B4

Let $A$ be a $2n \times 2n$ matrix, with entries chosen independently at random. Every entry is chosen to be 0 or 1, each with probability $1/2$. Find the expected value of $\det(A - A^t)$ (as a function of $n$), where $A^t$ is the transpose of $A$. 
Find all functions $f$ from the interval $(1, \infty)$ to $(1, \infty)$ with the following property:

if $x, y \in (1, \infty)$ and $x^2 \leq y \leq x^3$, then $(f(x))^2 \leq f(y) \leq (f(x))^3$.
Problem B6

Evaluate

\[ \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{n=0}^{\infty} \frac{1}{k2^n + 1}. \]